0 Preliminaries

0.1 Contents

0.2 Basic Graph Theory

A vertex is an object, usually a point represented by a number. We denote a set of m vertices by $V = \{1, \ldots, m\}$. An edge is a set of vertices (not necessarily just two). A set of n edges is denoted $\mathcal{E} = \{E_1, \ldots, E_n\}$. A graph is a pair $\mathcal{G} = (V, \mathcal{E})$ with $E \subseteq V \ \forall E \in \mathcal{E}$. A graph is complete if every set of vertices that could be an edge is in $\mathcal{E}$. A k-edge is an edge consisting of k vertices, and k-graph is a graph where all edges are k-edges.

A graph is c-colourable if we can colour the vertices in V with c colours so no edge $E \in \mathcal{E}$ is monochromatic (all vertices $v \in E$ have the same colour).

For $d \geq 2$, a d-cycle in $\mathcal{G} = (V, \mathcal{E})$ is a sequence $E_1, \ldots, E_d$ of distinct edges in $\mathcal{E}$ where we can find distinct vertices $v_i \in E_i \cup E_{i+1} \ \forall \ i \ \in \ \{1, \ \ldots, \ d\}$ with $E_{d+1} = E_1$. There is no such thing as a 1-cycle.

0.3 Partitions and Their Preservation

A graph $\mathcal{G} = (V, \mathcal{E})$ is called p-partite if there is a partition $V = V_1 \cup \cdots \cup V_p$ such that $|E \cup V_j| \leq 1 \ \forall E \in \mathcal{E}, j \in \{1, \ldots, p\}$. For these theorems simply knowing that a graph is p-partite is not enough – we want to specify the p-partition of a particular graph. We will write $\mathcal{G} = (\langle V_j \rangle, \mathcal{E}, p)$ to mean $\mathcal{G} = (\bigcup_{j=0}^{p} V_j, \mathcal{E})$ with the partition $V_1 \cup \cdots \cup V_p$.

Knowing the p-partition of a graph $\mathcal{G} = (\langle V_j \rangle, \mathcal{E}, p)$, we need to be able to preserve it under certain operations. Let $\mathcal{H} = (\langle W_j \rangle, \mathcal{F}, q)$.

- A map $b : \mathcal{G} \to \mathcal{H}$ must have $p = q$ and satisfy $b(v) \in W_j \ \forall v \in V_j$.
- The graphs $\mathcal{G}$ and $\mathcal{H}$ are isomorphic iff there is a (partition preserving) bijection $B : V \to W$ with $B(V_j) = W_j \ \forall j$ and $B(E) \in \mathcal{F}$ iff $E \in \mathcal{E}$.
- The graph $\mathcal{G}$ is a subgraph of $\mathcal{H}$ iff $p = q$, $V_j \subseteq W_j \ \forall j$ and $\mathcal{E} \subseteq \mathcal{F}$.
- A copy of $\mathcal{G}$ in $\mathcal{H}$ is a subgraph of $\mathcal{H}$ isomorphic to $\mathcal{G}$.

Note that we do not allow permution of the partitions – a subgraph of a p-partite graph will also have p partitions (although some may be empty). Similarly, if $\mathcal{H}$ consists of of n copies of $\mathcal{G}$ then $\mathcal{H}$ has the same number of partitions as $\mathcal{G}$, each containing n copies of the corresponding partition in $\mathcal{G}$.

I cannot emphasise enough how important the concept of preserving partitions is to constructing sparse Ramsey graphs and proving they have the desired properties. In particular, note the following construction.

Let S be a graph with l vertices. This is l-partite with one vertex x_j in each partition x_j. An edge $v = \{\mathrm{x}_{p_1}, \ldots, \mathrm{x}_{p_k}\}$ spans partitions $x_{p_1}, \ldots, x_{p_k}$. After any combination of (partition preserving) copying and mapping, the graph is still l-partite, each copy of the edge v spans partitions $x_{p_1}, \ldots, x_{p_k}$, and each copy of an edge or set of edges in S has at most one vertex in each partition.

0.4 Ramsey Theory

Theorem 1 (Ramsey on k-graphs).

Denote a complete k-graph with r vertices by $\mathcal{G}_k(r)$. Then for all positive integers c and k there is a function $\mathrm{b}_k : r \to s$ such that given r, any c-colouring of the graph $\mathcal{G}_k(s)$ has a monochromatic subset $\mathcal{G}_k(r)$.

Proof. (Induction on k)

The case $k = 1$ is trivial.

For $k \geq 2$, let $\mathrm{b}_{k-1}^{c(r-1)+1}(r) = \overbrace{1 + \mathrm{b}_{k-1}(\ldots(1 + \mathrm{b}_{k-1}}^{c(r-1)+1}(r))\ldots)$ and assert that $\mathrm{b}_k(r) \leq \mathrm{b}_{k-1}^{c(r-1)+1}(r)$. Let $a(E)$ be a c-colouring of $\mathcal{G}_k(\mathrm{b}_{k-1}^{c(r-1)+1}(r))$

Consider the graph $\mathcal{H}^0 = \mathcal{G}_{k-1}(\mathrm{b}_{k-1}^{c(r-1)+1}(r))$ with the same vertices as $\mathcal{G}_k(\mathrm{b}_{k-1}^{c(r-1)+1}(r))$. Choose a vertex of v_1 of $\mathcal{H}^0$ and consider the c-colouring of $\mathcal{H}^0 - \{v_1\}$ induced by $a'(E) = a(E \cup \{v_1\})$. By induction this has a monochromatic subset $\mathcal{H}^1 = \mathcal{G}_{k-1}(\mathrm{b}_{k-1}^{c(r-1)+1-1}(r))$ with colour a_1. Repeat to give a sequence of vertices $v_1, \ldots, v_{c(r-1)+1}$ and colours $a_1, \ldots, a_{c(r-1)+1}$.

There are r elements of $a_1, \ldots, a_{c(r-1)+1}$ sharing the same colour. The vertices corresponding to these form a monochromatic graph $\mathcal{G}_k(r)$.

$\square$

Theorem 2 (Ramsey on k-partite k-graphs).

Let $\mathcal{G}_k(r_1, \ldots, r_k)$ represent a complete k-partite k-graph with r_i vertices in the ith partition. Then for all positive integers c and k there is a function $\mathrm{b}_k : r_1, \ldots, r_k \to s_1, \ldots, s_k$ such that given $r_1, \ldots, r_k$, any c-colouring of the graph $\mathcal{G}_k(s_1, \ldots, s_k)$ has a monochromatic subset $\mathcal{G}_k(r_1, \ldots, r_k)$.

Proof. (Induction on k)

The case $k = 1$ is trivial.

For $k \geq 2$, let $\mathrm{b}_{k-1}^{c(r_k-1)+1}(r_1, \ldots, r_{k-1}) = \overbrace{\mathrm{b}_{k-1}(\ldots \mathrm{b}_{k-1}}^{c(r_k-1)+1}(r_1, \ldots, r_{k-1})\ldots)$ and assert that $\mathrm{b}_k(r_1, \ldots, r_k) \leq \mathrm{b}_{k-1}^{c(r_k-1)+1}(r_1, \ldots, r_{k-1}), c(r_k - 1) + 1$. Let $a(E)$ be a c-colouring of $\mathcal{G}_k(\mathrm{b}_{k-1}^{c(r_k-1)+1}(r_1, \ldots, r_{k-1}), c(r_k - 1) + 1)$ and write the set of vertices in the kth partion as $\{v_1, \ldots, v_{c(r_k-1)+1}\}$,

Consider the c-colouring of $\mathcal{H}^0 = \mathcal{G}_{k-1}(\mathrm{b}_{k-1}^{c(r_k-1)+1}(r_1, \ldots, r_{k-1}))$ induced by $a'(E) = a(E \cup \{v_1\})$. By induction this contains a monochromatic subset $\mathcal{H}^1 = \mathcal{G}_{k-1}(\mathrm{b}_{k-1}^{c(r_k-1)+1-1}(r_1, \ldots, r_{k-1}))$ with colour a_1. Repeat to give $\mathcal{H}^{c(r_k-1)+1} = \mathcal{G}_{k-1}(r_1, \ldots, r_{k-1})$ and a sequence of colours $a_1, \ldots, a_{c(r_k-1)+1}$.

There are r_k elements of $a_1, \ldots, a_{c(r_k-1)+1}$ sharing the same colour. The vertices in the kth partition corresponding to these, and the vertices in $\mathcal{H}^{c(r_k-1)+1}$ form a monochromatic subset $\mathcal{G}_k(r_1, \ldots, r_k)$.

$\square$

1 Vertex Form

1.1 Main Theorem

This is the method we use to construct sparse Ramsey graphs, in its most basic form. There is an example of how we use the method in section 1.2 and a list of variables it uses in appendix A. Finally, let me reemphasise the importance of partition preservation which was discussed in section 0.3.

Theorem 3 (Sparse Ramsey - Vertex Form).
For positive integers c, D and k, we can construct a k-graph $\mathcal{G}$ such that

1. *$\mathcal{G}$ is not c-colourable.*
2. *For $D \neq 1$ write $\mathcal{G} = (V, \mathcal{E})$ and choose $E_1 \neq E_2$ in $\mathcal{E}$. Then E_1 and E_2 have at most one vertex in common.*

   [This statement is equivalent to saying that $\mathcal{G}$ has no 2-cycles. I have left it separate from condition 3 because it needs to be proved separately.]
3. *$\mathcal{G}$ has no d-cycles for $d \leq D$.*

Proof. (Induction on D)
When $D = 1$ there are no restrictions on cycles and so we only need to construct $\mathcal{G}$ satisfying condition 1. We can do this by Ramsey as follows. Let $\mathcal{G}$ be a complete k-graph with $k(c-1)+1$ edges. For any c-colouring there are k vertices with the same colour. Since $\mathcal{G}$ is complete, there is an edge consisting of these vertices. Thus $\mathcal{G}$ is not c-colourable.

Suppose the theorem is true for $D-1$. Then the following construction gives a graph $\mathcal{G}$ satisfying the conditions.

1. Using the same method that we used in the case $D = 1$, find a graph $\mathcal{G}^\star = (V^\star, \mathcal{E}^\star)$ satisfying condition 1. Write $V^\star = \{v_1^\star, \ldots, v_m^\star\}$ and $\mathcal{E}^\star = \{E_1^\star, \ldots, E_n^\star\}$. Now choose partitions $V_j^\star = \{v_j^\star\}$ to give the m-partite graph $\mathcal{G}^\star = (\langle V_j^\star \rangle, \mathcal{E}^\star, m)$.
2. Let $\mathcal{G}^0 = (\langle V_j^0 \rangle, \mathcal{E}^0, m)$ consist of n copies of $\mathcal{G}^\star$, each copy corresponding to an edge $E_j^\star \in \mathcal{E}^\star$. Since we are preserving partitions this gives us $V_j^0 = \{\text{all copies of } V_j^\star\}$ and $\mathcal{E}^0 = \{\text{all copies of } \mathcal{E}^\star\}$.

   For each copy of $\mathcal{G}^\star$ in $\mathcal{G}^0$, remove all edges E except the edge which the copy corresponds to. Now remove all $v \in V^0$ not in some $E \in \mathcal{E}^0$. [Removing vertices is not strictly necessary as their presence doesn't affect the properties of the graph, but it makes the process cleaner.]

3. Suppose we have constructed the graph $\mathcal{G}^{i-1} = (\langle V_j^{i-1} \rangle, \mathcal{E}^{i-1}, m)$. We will now construct the graph $\mathcal{G}^i$.

   Consider the partition V_i^{i-1} and let $K = |V_i^{i-1}|$. By induction construct a K-graph $\mathcal{H}^i = (W^i, \mathcal{F}^i)$, such that $\mathcal{H}^i$ is not c-colourable and has no d-cycles when $d \leq D - 1$.

   Take $|\mathcal{F}^i|$ copies of $\mathcal{G}^{i-1}$, each copy corresponding to an edge $F \in \mathcal{F}^i$ of $\mathcal{H}^i$. Given F, define a partition preserving bijection b_F^i which maps vertices $v \in V_i^{i-1}$ in the copy corresponding to F into F itself, and acts as the identity operator otherwise. Now define B_F^i to be a bijection such that $B_F^i(E) = \{b_F^i(v) : v \in E\}$.

   Now let $V_j^i = \bigcup_{v \in V_j^{i-1}, F \in \mathcal{F}} b_F(v) \qquad V^i = \bigcup_{j=1}^m V_j^i$
   $\mathcal{E}^i = \bigcup_{E \in \mathcal{E}^{i-1}, F \in \mathcal{F}} B_F(E)$

   This gives a graph $\mathcal{G}^i = (\langle V_j^i \rangle, \mathcal{E}^i, m)$. Note that $\mathcal{E}^i$ does not contain any edges of $\mathcal{F}^i$ – we're only using them to guide the mapping.

4. The graph $\mathcal{G} = \mathcal{G}^m$ is the desired graph.

We have claimed that $\mathcal{G}$ is the desired graph. Certainly $\mathcal{G}$ is a k-graph, so we only need to check that it satisfies the conditions.

1. *$\mathcal{G}$ is not c-colourable.*

   Suppose otherwise. Then there is a c-colouring of $\mathcal{G} = \mathcal{G}^m$. Apply the following inductively.

   Given a c-colouring of $\mathcal{G}^i$, consider the c-colouring this induces on $\mathcal{H}^i$. By construction this is not c-colourable, so we can find a monochromatic edge F. The copy of $\mathcal{G}^{i-1}$ corresponding to F has a monochromatic ith partition. Now consider $\mathcal{G}^{i-1}$ with the induced c-colouring.

   This gives a graph $\mathcal{G}^0$ in which every partition is monochromatically coloured. Colour $\mathcal{G}^\star$ so that each vertex is the same colour as the corresponding partition. By construction, $\mathcal{G}^\star$ is not c-colourable, so this gives a monochromatic edge. The corresponding edge in $\mathcal{G}^0 \subseteq \mathcal{G}$ is also monochromatic which is a contradiction.

2. *For $D \neq 1$ write $\mathcal{G} = (V, \mathcal{E})$ and choose $E_1 \neq E_2$ in $\mathcal{E}$. Then E_1 and E_2 have at most one vertex in common.*

   We will prove this by induction on i. The edges in $\mathcal{G}^0$ are disjoint, so $i = 0$ is trivially true. Suppose the claim is true for $i - 1$ and let E_1^i, E_2^i be edges of $\mathcal{G}^i = (\langle V_j^i \rangle, \mathcal{E}^i, m)$. Write $E_j^i = b_{F_j}^i(E_j^{i-1})$. If $F_1 = F_2$, then the claim is true by induction.

Otherwise $F_1 \neq F_2$ and all common vertices are in partition i. Now, because all edges in $\mathcal{G}^i$ are copies of edges in $\mathcal{G}^\star$, and $\mathcal{G}^\star$ has at most one vertex in each partition, we know that E_1^i and E_2^i have at most one vertex in each partition. Thus they intersect at most once for each partition they have in common.

Note that because E_j^i contains the vertices of F_j, the common vertex of E_1^i and E_2^i will be the same as the common vertex of F_1 and F_2.

3. *$\mathcal{G}$ has no d-cycles for $d \leq D$.*

   Suppose for contradiction that the graph $\mathcal{G}$ has a d-cycle for some $d \leq D$. We know that $\mathcal{G}^0$ is disjoint, and so has no d-cycle, or indeed any cycle. Thus there is an i such that $\mathcal{G}^{i-1}$ has no d-cycle with $d \leq D$ but $\mathcal{G}^i$ does. Note that $\mathcal{G}^i$ does not have a d-cycle with $d \leq 2$ because of condition 2, and the fact that d-cycles with $d \leq 1$ do not exist.

   Thus we have a d-cycle with $3 \leq d \leq D$, with distinct edges $[E_1^i, \ldots, E_d^i]$ and distinct vertices $v_j \in E_j^i \cap E_{j+1}^i$ where $E_{d+1}^i = E_1^i$. Write each edge as $E_j^i = b_{F_j}^i(E_j^{i-1})$ and consider how many different F_j's there are.

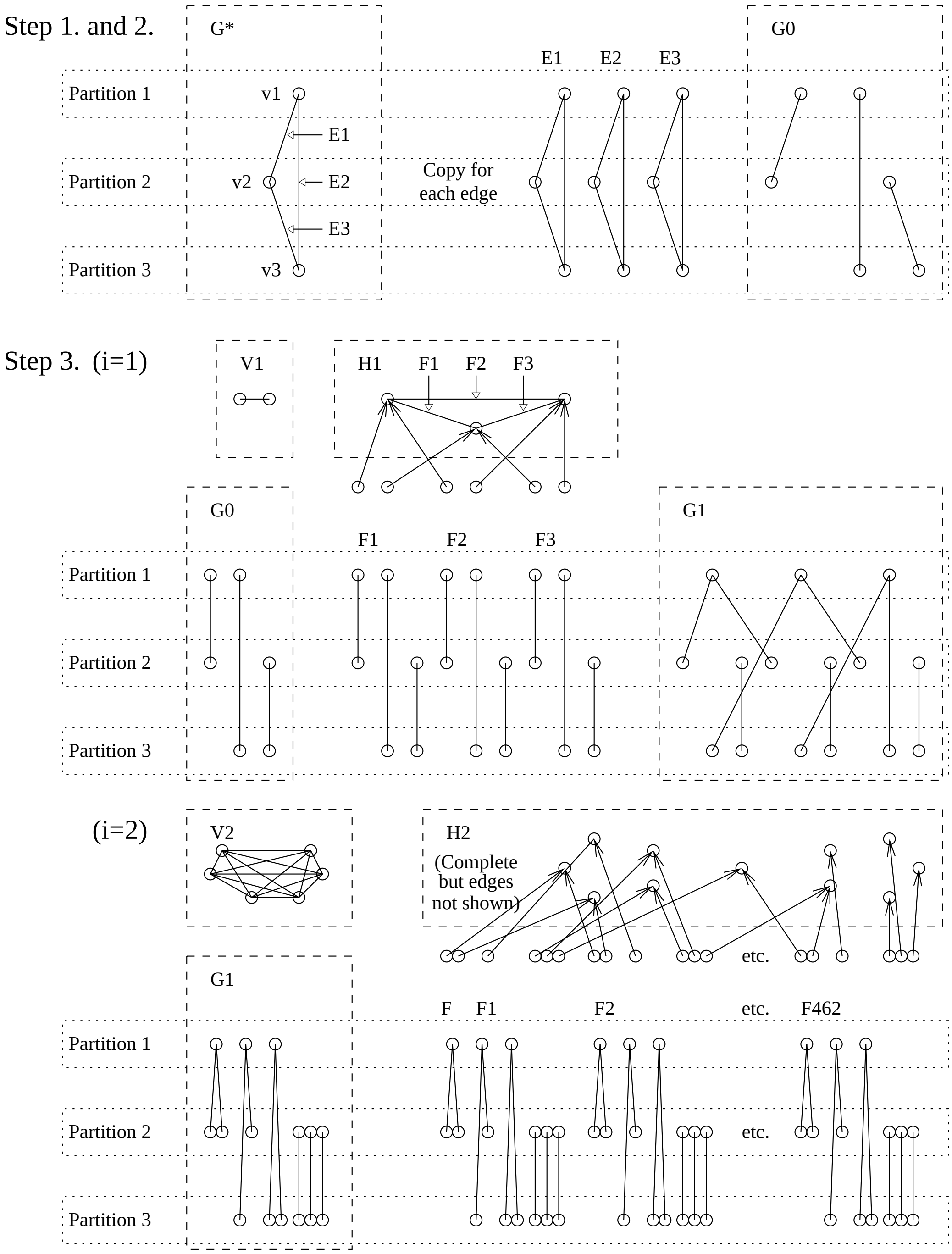
   - If there is only one (i.e. $F_1 = F_2 = \ldots = F_d$), then $[E_1^{i-1}, \ldots, E_d^{i-1}]$ forms a d-cycle in in $\mathcal{G}^{i-1}$, which is a contradiction.
   - If there are two, then w.l.o.g. let $F_1 \neq F_2$ and $F_i \neq F_{i+1}$ with $i \neq 1$. Condition 2 tells us that if two adjacent edges have different F_j's, then they have a single common vertex which is also the single common vertex of the F_j's. As there are only two different F_j's, the vertex between F_1 and F_2 must be the same as the vertex between F_i and F_{i+1}, which is a contradiction as we have stated that they are distinct.
   - If there are p different F_j's, where $3 \leq p < D$, then the set of different F_j's forms a p-cycle in $\mathcal{H}^i$, with $p \leq D - 1$. But $\mathcal{H}^i$ contains no such cycle by construction, so this is a contradiction.
   - If there are D, then every F_j is different. Condition 2 tells us that if two adjacent edges have different F_j's, then the edges have a single common vertex in partition i. Now, because all edges in $\mathcal{G}^i$ are copies of edges in $\mathcal{G}^\star$, and $\mathcal{G}^\star$ only has one vertex in each partition, every edge in the cycle has only one vertex in partition i. Thus the vertices v_j are all the same vertex in partition i, contradicting the statement that they are distinct.

This completes the proof.

□

1.2 Example

The following is the start of the construction of the sparse Ramsey graph given in theorem 3, with $c = 2$, $D = 3$ and $k = 2$. I stop in stage 3 when $i = 2$, as drawing over 4000 vertices seems somewhat wasteful.



2 Edge Form

2.1 Main Theorem

This theorem generalises the main theorem of paper [2] from graphs to k-graphs. I have considerably reworked the proof so that it is as similar in form to that of theorem 3 as possible, preserving in particular the same structure and variables. Thus, if you have any problems refer back to the corresponding part of theorem 3. There is an example of the use of this method in section 2.2 and a list of variables used in appendix A.

Theorem 4 (Sparse Ramsey - Edge Form).
Let c, D and k be positive integers and $R = (x, U)$ be a k-graph. Define '$\mathcal{G}$ on S' to mean that $\mathcal{G}$ is a graph whose vertices are edges of the k-graph S and each of whose edges $E \in \mathcal{E}$ induces a graph isomorphic to R (unless we state otherwise). Now we can construct $\mathcal{G}$ on S such that

1. *If R is k-partite then S is k-partite.*

2. *$\mathcal{G}$ is not c-colourable.*

3. *For $D \neq 1$ write $\mathcal{G} = (V, \mathcal{E})$ and choose $E_1 \neq E_2$ in $\mathcal{E}$. Then E_1 and E_2 have at most one vertex in common.*

   [This statement is equivalent to saying that $\mathcal{G}$ has no 2-cycles. I have left it separate because it needs to be proved separately. In their proof of this theorem, Nešetřil and Rödl used a stronger statement instead – this is unnecessary, and moreover is false for $k \neq 2$.]

4. *$\mathcal{G}$ has no d-cycles for $d \leq D$.*

Proof. (Induction on D)
When $D = 1$ there are no restrictions on cycles and so we only need to construct $\mathcal{G}$ on S satisfying conditions 1 and 2. We can do this by Ramsey.

If R is not k-partite, let R^c be the complete k-graph on x and apply theorem 1. This gives $S = (y, V)$ with a monochromatic copy of R^c and thus a monochromatic copy of $R \subseteq R^c$. Thus $\mathcal{G} = (V, \mathcal{E})$ with $\mathcal{E} = \{$set of all subsets of S isomorphic to $R\}$ satisfies the conditions.

If R is k-partite, write $R = (\langle x_j \rangle, U, k)$. Let R^c be the complete k-partite k-graph on $\langle x_j \rangle$ and apply theorem 2. This gives a k-partite k-graph $S = (\langle y_j \rangle, V, k)$ containing a monochromatic copy of R^c and thus a monochromatic copy of $R \subseteq R^c$. Thus $\mathcal{G} = (V, \mathcal{E})$ with $\mathcal{E} = \{$set of all subsets of S isomorphic to $R\}$ satisfies the conditions.

Suppose the theorem is true for $D-1$. Then the following construction gives graphs $\mathcal{G}$ on S satisfying the conditions.

1. Using the same method that we used in the case $D=1$, find graphs $\mathcal{G}^\star = (V^\star, \mathcal{E}^\star)$ on $S^\star = (y^\star, V^\star)$ satisfying both conditions 1 and 2. Write $y^\star = \{\mathrm{y}_1^\star, \ldots, \mathrm{y}_l^\star\}$, $V^\star = \{v_1^\star, \ldots, v_m^\star\}$ and $\mathcal{E}^\star = \{E_1^\star, \ldots, E_n^\star\}$. Now choose partitions $V_j^\star = \{v_j^\star\}$ to give $\mathcal{G}^\star = (\langle V_j^\star \rangle, \mathcal{E}^\star, m)$ m-partite and choose partitions $y_j^\star = \{\mathrm{y}_j^\star\}$ to give $S^\star = (\langle y_j^\star \rangle, V^\star, l)$ l-partite.

2. Let $\mathcal{G}^0 = (\langle V_j^0 \rangle, \mathcal{E}^0, m)$ on $S^0 = (\langle y_j^0 \rangle, V^0, l)$ consist of n copies of $\mathcal{G}^\star$ on $S^\star$, each copy corresponding to an edge $E_j^\star \in \mathcal{E}^\star$. Since we are preserving partitions, this gives us $y_j^0 = \{$all copies of $y_j^\star\}$, $V_j^0 = \{$all copies of $V_j^\star\}$ and $\mathcal{E}^0 = \{$all copies of $\mathcal{E}^\star\}$.

   For each copy of $\mathcal{G}^\star$ in $\mathcal{G}^0$, remove all edges E except the edge which the copy corresponds to. Now remove all $v \in V^0$ not in some $E \in \mathcal{E}^0$, and then all $\mathrm{y} \in y^0$ not in some $v \in V^0$. [Removing v and y is not strictly necessary, as their presence doesn't affect the properties of the graph, but their removal makes the process cleaner and allows corollary 6.]

3. Suppose we have constructed the graphs $\mathcal{G}^{i-1} = (\langle V_j^{i-1} \rangle, \mathcal{E}^{i-1}, m)$ on $S^{i-1} = (\langle y_j^{i-1} \rangle, V^{i-1}, l)$. We will now construct the graphs $\mathcal{G}^i$ on S^i.

   Consider the partition V_i^{i-1}, containing k partitions $y_{p_1}^{i-1}, \ldots, y_{p_k}^{i-1}$ of y^{i-1}. These form a k-partite k-graph $R^i = (\langle y_{p_j}^{i-1} \rangle, V_i^{i-1}, k)$. By induction, construct $\mathcal{H}^i = (W^i, \mathcal{F}^i)$ on a k-partite k-graph $T^i = (\langle z_{p_j}^i \rangle, W^i, k)$ each of whose edges $F \in \mathcal{F}^i$ induces a graph isomorphic to R^i, such that $\mathcal{H}^i$ is not c-colourable, obeys condition 3 if $D-1 \neq 1$, and has no d-cycles when $d \leq D-1$.

   Take $|\mathcal{F}^i|$ copies of $\mathcal{G}^{i-1}$ on S^{i-1}, each copy corresponding to an edge $F \in \mathcal{F}^i$ of $\mathcal{H}^i$. Given F, define a partition preserving bijection b_F^i which, for $j \in \{p_1, \ldots, p_k\}$, maps vertices $\mathrm{y} \in y_j^{i-1}$ in the copy corresponding to F to $\mathrm{z} \in z_j^i \cap (\bigcup_{w \in F} w)$, and acts as the identity operator otherwise. Now define b_F^i to be a bijection such that $b_F^i(v) = \{\mathrm{b}_F^i(\mathrm{x}) : \mathrm{x} \in v\}$ and B_F^i to be a bijection such that $B_F^i(E) = \{b_F^i(v) : v \in E\}$.

   Now let
   $$\begin{aligned} y_j^i &= \textstyle\bigcup_{\mathrm{y} \in y_j^{i-1}, F \in \mathcal{F}} \mathrm{b}_F(\mathrm{y}) & y^i &= \textstyle\bigcup_{j=1}^l y_j^i \\ V_j^i &= \textstyle\bigcup_{v \in V_j^{i-1}, F \in \mathcal{F}} b_F(v) & V^i &= \textstyle\bigcup_{j=1}^m V_j^i \\ \mathcal{E}^i &= \textstyle\bigcup_{E \in \mathcal{E}^{i-1}, F \in \mathcal{F}} B_F(E) \end{aligned}$$
   This gives graphs $\mathcal{G}^i = (\langle V_j^i \rangle, \mathcal{E}^i, m)$ on $S^i = (\langle y_j^i \rangle, V^i, l)$

4. The graph $\mathcal{G} = \mathcal{G}^m$ on $S = S^m$ is the desired graph.

We have claimed that $\mathcal{G}$ on S is the desired graph. Certainly $\mathcal{G}$ is a graph whose vertices are edges of the k-graph S and whose edges $E \in \mathcal{E}$ are isomorphic to R. So we need to check that it satisfies the conditions.

1. *If R is k-partite then S is k-partite.*

   Let R be k-partite. Now S is k-partite when $D = 1$ and similarly $S^\star$ is k-partite when $D \neq 1$. But $S^\star$ is also l-partite, and every k-partition can be formed by joining l-partitions. We claim S is k-partite with each k-partition formed by joining the same l-partitions as $S^\star$.

   Suppose for contradiction that this is not the case. Then there is an edge in S which has more than one vertex in some k-partitions. But this edge is a copy of an edge in $S^\star$, which because of partition preservation must also have more than one vertex in the corresponding k-partition. Thus $S^\star$ is not k-partite, which is a contradiction.

2. *$\mathcal{G}$ is not c-colourable.*

   Consider an c-colouring on $\mathcal{G} = \mathcal{G}^n$ and apply the following inductively.

   Given a c-colouring of $\mathcal{G}^i$, consider the c-colouring this induces on $\mathcal{H}^i$. By construction this is not c-colourable, so we can find a monochromatic edge F. The copy of $\mathcal{G}^{i-1}$ corresponding to F has a monochromatic ith partition. Now consider $\mathcal{G}^{i-1}$ with the induced c-colouring.

   This gives a graph $\mathcal{G}^0$ in which every partition is monochromatically coloured. Colour $\mathcal{G}^\star$ so that each vertex v is the same colour as the corresponding partition. By construction, $\mathcal{G}^\star$ is not c-colourable, so this gives a monochromatic edge. The corresponding edge in $\mathcal{G}^0 \subseteq \mathcal{G}$ is also monochromatic and so $\mathcal{G}$ is not c-colourable.

3. *For $D \neq 1$ write $\mathcal{G} = (V, \mathcal{E})$ and choose $E_1 \neq E_2$ in $\mathcal{E}$. Then E_1 and E_2 have at most one vertex in common.*

   We will prove this by induction on i. The edges in $\mathcal{G}^0$ are disjoint, so $i = 0$ is trivially true. Suppose the claim is true for $i-1$ and let E_1^i, E_2^i be edges of $\mathcal{G}^i = (\langle V_j^i \rangle, \mathcal{E}^i, m)$. Write $E_j^i = b_{F_j}^i(E_j^{i-1})$. If $F_1 = F_2$, then the claim is true by induction.

   Otherwise $F_1 \neq F_2$ and all common vertices are in partition i. Now, because all edges in $\mathcal{G}^i$ are copies of edges in $\mathcal{G}^\star$, and $\mathcal{G}^\star$ has at most one vertex in each partition, we know that E_1^i and E_2^i have at most one vertex in each partition. Thus they intersect at most once for each partition they have in common.

   Note that because E_j^i contains the vertices of F_j, the common vertex of E_1^i and E_2^i will be the same as the common vertex of F_1 and F_2.

4. *$\mathcal{G}$ has no d-cycles for $d \leq D$.*

   Suppose for contradiction that the graph $\mathcal{G}$ has a d-cycle where $d \leq D$. We know that $\mathcal{G}^0$ is disjoint, and so has no d-cycle, or indeed any cycle. Thus there is an i such that $\mathcal{G}^{i-1}$ has no d-cycle with $d \leq D$ but $\mathcal{G}^i$ does. Note that $\mathcal{G}^i$ does not have $d \leq 2$ because of condition 3, and the fact that d-cycles with $d \leq 1$ do not exist.

   Thus we have a d-cycle with $3 \leq d \leq D$, with distinct edges $[E_1^i, \ldots, E_d^i]$ and distinct vertices $v_j \in E_j^i \cap E_{j+1}^i$ where $E_{d+1}^i = E_1^i$. Write each edge as $E_j^i = B_{F_j}^i(E_j^{i-1})$ and consider how many different F_j there are.

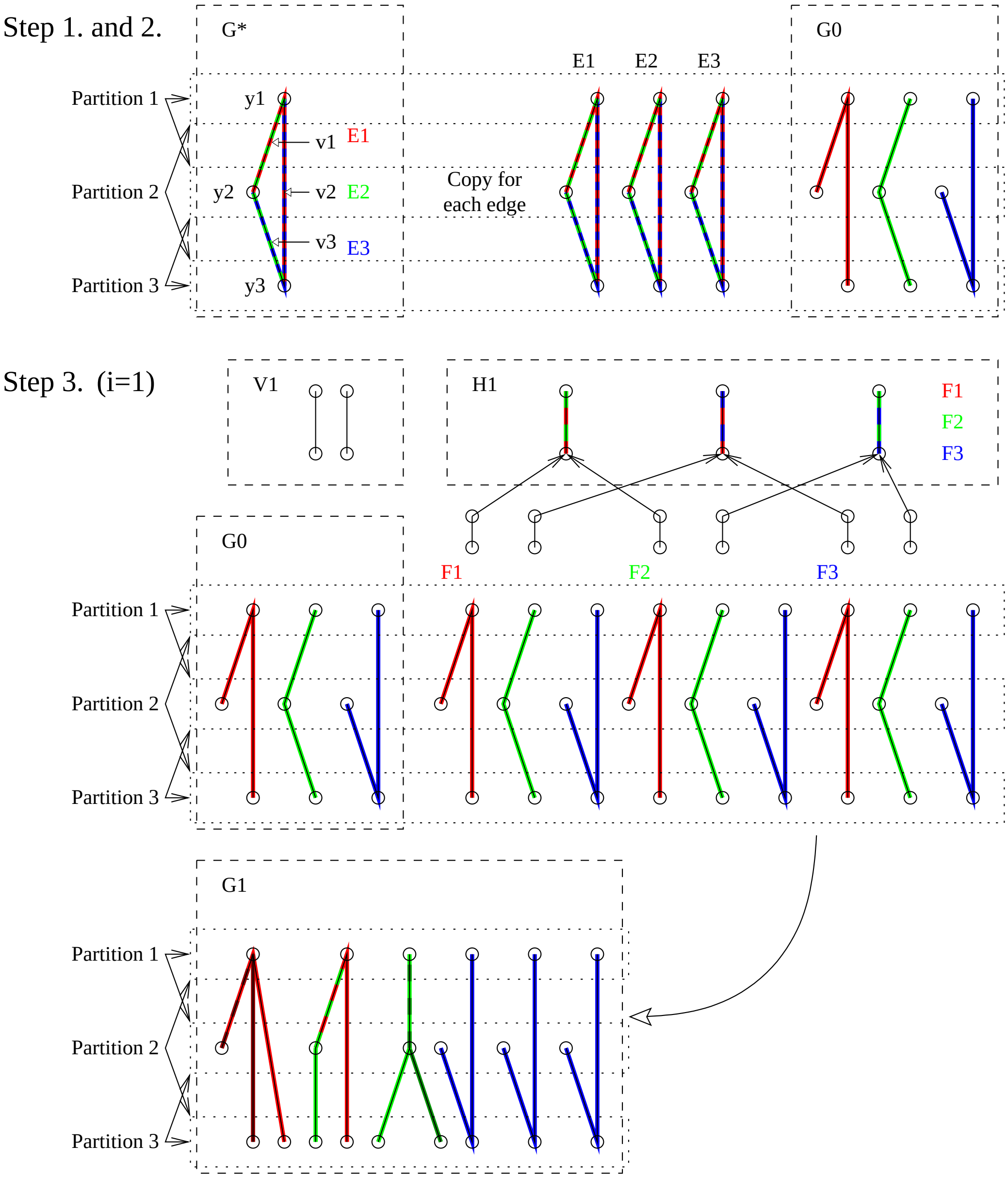
   - If there is only one (i.e. $F_1 = F_2 = \ldots = F_d$), then $[E_1^{i-1}, \ldots, E_d^{i-1}]$ forms a d-cycle in in $\mathcal{G}^{i-1}$, which is a contradiction.
   - If there are two, then w.l.o.g. let $F_1 \neq F_2$ and $F_i \neq F_{i+1}$ with $i \neq 1$. Condition 3 tells us that if two adjacent edges have different F_j's, then they have a single common vertex which is also the single common vertex of the F_j's. As there are only two different F_j's, the vertex between F_1 and F_2 must be the same as the vertex between F_i and F_{i+1}, which is a contradiction as we have stated that they are distinct.
   - If there are p different F_j's, where $3 \leq p < D$, then the set of different F_j's forms a p-cycle in $\mathcal{H}^i$, with $p \leq D - 1$. But $\mathcal{H}^i$ contains no such cycle by construction, so this is a contradiction.
   - If there are D, then every F_j is different. Condition 3 tells us that if two adjacent edges have different F_j's, then the edges have a single common vertex in partition i. Now, because all edges in $\mathcal{G}^i$ are copies of edges in $\mathcal{G}^\star$, and $\mathcal{G}^\star$ only has one vertex in each partition, every edge in the cycle has only one vertex in partition i. Thus the vertices v_j are all the same vertex in partition i, contradicting the statement that they are distinct.

This completes the proof.

□

2.2 Example

The following is the start of the construction of the sparse Ramsey graph given in theorem 4, with $c = 2$, $D = 3$, $k = 2$ and R consisting of two edges joined at one end. I have used colours to distinguish elements of $\mathcal{E}$ – a line with a dashed colour is part of two edges. I stop in stage 3 because the construction is too hard to draw when $i = 2, 3$.



2.3 Corollaries

For completeness, I have reworked the corollary in paper [2] into the form used in theorem 4.

Corollary 5.

Let c, D and k be positive integers and $R = (x, U)$ be a complete k-graph. Then we can construct $\mathcal{G} = (V, \mathcal{E})$ on $S = (y, V)$ such all the conditions in theorem 4 are satisfied, and $\mathcal{E} = \{$all copies of R in $S\}$.

Proof.

If we follow the construction in theorem 4 with $R = (x, U)$ a complete k-graph, then we get $\mathcal{G} = (V, \mathcal{E})$ on $S = (y, V)$ satisfying all the conditions. We claim that $\mathcal{E} = \{\text{all copies of } R \text{ in } S\}$ and $\mathcal{E}^i = \{\text{all copies of } R \text{ in } S^i\}$.

We will prove this by induction on i. The edges in $\mathcal{G}^0$ are disjoint, and we have removed all vertices not in some edge $E \in \mathcal{E}^0$, so when $i = 0$, the claim must be true. Suppose the claim is true for $i - 1$ and suppose for contradiction that it is not true for i. Then there is some $E =$ a copy of R in S^i, such that $E \notin \mathcal{E}^i$.

If E is completely contained in one copy of S^{i-1}, then the claim cannot be true for $i-1$, which is a contradiction. Otherwise, E has two edges v_1, v_2 such that each is in a copy of S^{i-1} that the other is not in. In turn, these contain vertices $\mathrm{y}_1, \mathrm{y}_2$ such that each is in a copy of S^{i-1} that the other is not in. But then y_1 and y_2 are not in the same edge as each other, contradicting the claim that E is copy of a complete graph.

□

It is also conjectured that for any R, we can construct $\mathcal{G} = (V, \mathcal{E})$ on $S = (y, V)$ such that all the conditions in theorem 4 are satisfied, and $\mathcal{E} = \{\text{all copies of } R \text{ in } S\}$.

3 Space Form

3.1 Main Theorem

The technique described in sections 1 and 2 can be applied more generally than just on graphs. The theorem below is an example of this – applying the technique to Graham-Rothschild.

It is assumed that you are comfortable with Graham-Rothschild and therefore with manipulating combinatorial spaces. The proof is deliberately light on the exact details of how you apply it, because I found that the proof in paper [3] tended to get bogged down in explanations.

Note that we are not using the notation developed in the previous sections, although I have tried to preserve some common elements. Again you can find a list of variables in appendix A.

Theorem 6 (Sparse Ramsey - Space Form).

Given an e-space E containing a family of a-spaces $\mathcal{E}$, and integers c and $b \geq a$, we can find a g-space G containing a family of a-spaces $\mathcal{G}$ such that

1. *Given a c-colouring of the b-spaces contained in $\mathcal{G}$, we can find an e-space in G containing a family of a-spaces $\mathcal{E}' \subseteq \mathcal{G}$ such that $\mathcal{E}$' is isomorphic to $\mathcal{E}$ and the b-spaces contained in $\mathcal{E}$' are monochromatic.*

2. *Given $b \leq d \leq e$, if we can't find a d-space contained in $\mathcal{E}$, then we can't find a d-space contained in $\mathcal{G}$. [This is true for all d, but only interesting in these cases].*

Proof.

1. Use Graham-Rothschild to find an $g^\star$-space $G^\star$ such that, given an arbitrary c-colouring of the b-spaces in $G^\star$, we can find an e-space in which all b-spaces are monochromatic. Label the e-spaces in $G^\star$ as $N_1, \ldots, N_n$ and let every N_j contain a family of a-spaces isomorphic to $\mathcal{E}$. Let $\mathcal{G}^\star$ be the union of all a-spaces in N_j and note that $G^\star$ and $\mathcal{G}^\star$ satisfy condition 1. Label the b-spaces in $\mathcal{G}^\star$ as $M_1, \ldots, M_m$.

2. Define the projection of a space or set of spaces to be the structure in $G^\star$ isomorphic to the structure found in the first $g^\star$ dimensions of the space it is in. Construct a g^0-space G^0 which contains e-spaces $N'_1, \ldots, N'_n$ where the projection of N'_j is N_j and which are pairwise disjoint. Furthermore, let every N'_j contain a family of a-spaces isomorphic to $\mathcal{E}$, such that the projection of each a-space is an a-space in N_j. The union of these gives a family of a-spaces $\mathcal{G}^0$. Note that the projection of every b-space in $\mathcal{G}^0$ is a b-space in $\mathcal{G}^\star$.

3. Given G^{i-1}, let $L_1, \dots, L_l$ be the set of b-spaces whose projection is M_i. Use Graham-Rothschild to construct an h^i-space H^i containing a monochromatic line length l, where each vertex in H^i is a separate b-space. For each line in H^i, take a copy of G^{i-1} corresponding to that line and replace $L_1, \dots, L_l$ with the b-spaces in the line (you may have to adjust the b-spaces in the line so that they overlap correctly before you do so). Finally merge the first $g^\star$-dimensions of each copy together, which gives the g^i-space G^i. It has a family of a-spaces $\mathcal{G}^i$ corresponding to the union of all the a-spaces in each copy.

4. The $G = G^m$, and $\mathcal{G} = \mathcal{G}^m$ is the desired spaces and set of a-spaces.

We have claimed that G is the desired g-space. So we need to check that it satisfies the conditions.

1. *Given a c-colouring of the b-spaces contained in $\mathcal{G}$, we can find an e-space in G containing a family of a-spaces $\mathcal{E}' \subseteq \mathcal{G}$ such that $\mathcal{E}$' is isomorphic to $\mathcal{E}$ and the b-spaces contained in $\mathcal{E}$' are monochromatic.*

   Suppose we have an arbitrary c-colouring of G^m. Apply the following step inductively.

   We have a c-colouring of G^i. This induces a c-colouring on H^i, which contains a monochromatic line. The copy of G^{i-1} corresponding to this line has all b-spaces whose projection is M_i monochromatic. Consider the c-colouring on this copy of G^{i-1}.

   This gives G^0 where the colour of each b-space depends only on its projection. This induces a colouring of $G^\star$, which has an e-space satisfying condition 1. The corresponding e-space in $G^0 \subset G$ also satisfies condition 1.

2. *Given $b \leq d \leq e$, if we can't find a d-space contained in $\mathcal{E}$, then we can't find a d-space contained in $\mathcal{G}$.*

   G^0 certainly contains no d-space, so suppose for contradiction that G does. Then we can find i such that G^i contains a d-space, and G^{i-1} does not. Since the only overlaps between copies of g^{i-1} which can occur are b-spaces, this is a contradiction.

□

A Variable Names

I have summarised all the variables used in this paper below. The font generally signifies a relationship between sets and their elements. These are not intended to be understood without reference to the theorems.

c	the chromatic number of either a graph, of a set of b-spaces.
i	a variable numbering a sequence of graphs $\mathcal{G}^i$, or of g^i-spaces G^i.
j	a useful variable used for many purposes

The following are used only in sections 1 and 2.

$\mathrm{x} \in x$	$u \in U$		$R = (x, U)$		
$\mathrm{y} \in y$	$v \in V$	$E \in \mathcal{E}$	$S = (y, V)$	$\mathcal{G} = (V, \mathcal{E})$	$\mathcal{G}$ a k-graph
$\mathrm{z} \in z$	$w \in W$	$F \in \mathcal{F}$	$T = (z, W)$	$\mathcal{H} = (W, \mathcal{F})$	$\mathcal{H}$ a K-graph
$l = \lvert y^\star \rvert$	$m = \lvert V^\star \rvert$	$n = \lvert \mathcal{E}^\star \rvert$			
$\mathrm{b}_{\mathcal{F}}(\mathrm{y})$	$b_{\mathcal{F}}(v)$	$B_{\mathcal{F}}(E)$	Functions on y, v and E.		

d, D	the number of elements in a cycle and the girth of a graph.
p	a useful variable used mainly to describe partitions

The following are only used in section 3.

size of a space	$a \leq b \leq d \leq$	$e \;\leq$	$g^\star \;\leq$	$g^0 \;\leq \ldots \leq$	$g^m = g$	h^1	$\ldots \; h^m$
a space		E	$G^\star$	$G^0 \quad \ldots$	$G^m = G$	H^1	$\ldots \; H^m$
set of a-spaces		$\mathcal{E}$	$\mathcal{G}^\star$	$\mathcal{G}^0 \quad \ldots$	$\mathcal{G}^m = \mathcal{G}$		

set of b-spaces	$L_1, \ldots, L_l$	$M_1, \ldots, M_m$
set of e-spaces	$N_1, \ldots, N_n$	

B References

[1] Jarsolsav Nešetřil and Vojtěch Rödl, *A Short Proof of the Existence of Highly Chromatic Hypergraphs without Short Cycles*, Journal of Combinatorial Theory, Series B 27 225-227 (1979)

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[3] P. Frankl, R. L. Graham and V. Rödl, *Induced Restricted Ramsey Theorems for Spaces*, Journal of Combinatorial Theory, Series A 44, 120-128 (1987)