Utility Functions

If an investor with wealth w ranks changes in wealth A, B and C so they satisfy

Comparability	either $A > B$, $A < B$ or $A = B$
Transitivity	if $A > B$ and $B > C$, then $A > C$
Independance	$A = B \Rightarrow pA + (1-p)C = pB + (1-p)C$
Certainty equivalence	$A > B > C \Rightarrow \exists\, p \in (0,1) \quad pA + (1-p)C = B$

then a function $U(w)$ representing their utility of wealth can be constructed. This may or may not obay the following conditions

Non satiation	$U'(w) > 0$	People prefer more to less
Risk adverse	$U''(w) < 0$	Would reject a fair gamble
Risk neutral	$U''(w) = 0$	Indifferent to a fair gamble
Risk seeking	$U''(w) > 0$	Would accept a fair gamble

Typically, people are considered non-satiable and risk adverse

The certainty equivalent is how much someone would pay to avoid a fair gamble of $\pm x$. That is $U(c_w = w + c_x) = E\,U(w \pm x)$

The ratio of c_w to w is called absolute risk adversion $A(w) = {-U''(w)}/{U'(w)}$
It typically decreases with wealth due to the lessenning importance of the gamble.

The ratio of c_w/w to w is called relative risk adversion $R(w) = {-wU''(w)}/{U'(w)}$
It's relationship with wealth is unclear.

Simple utility functions include quadratic $U(w) = a + bw + cw^2 \equiv w + dw^2$, log $U(w) = \ln w$ and power $U(w) = {w^{\gamma}-1}/{\gamma}$. More complex ones include state dependant functions, where dropping below a level of wealth trigge different behaviour.

Stochastic Dominance

Consider portfolios A and B with cumulative probability functions of returns F_A and F_B.
A has absolute dominance over B if it provides better returns in all circumstances.
This is unlikely, so we consider stochastic dominance

First order stochastic dominance

$F_A(x) - F_B(x) \leq 0 \quad \forall x$ and $\exists x$ with the inequality strict.

An investor who prefers more to less will prefer A to B.

Second order stochastic dominance

$\int_{-\infty}^{x} F_A(y) - F_B(y)\, dy \leq 0 \quad \forall x$ and $\exists x$ with the inequality strict

A non-saitiable risk adverse investor will prefer A to B.

Third order stochastic dominance

$\int_{-\infty}^{x} \int_{-\infty}^{y} F_A(z) - F_B(z)\, dz dy \leq 0 \quad \forall x$ and $\exists x$ with the inequality strict

A non-saitiable risk adverse investor with decreasing absolute risk aversion will prefer A to B

Stochastic dominance allows us to make decisions without knowing much about the shape of an investors utility function.

Efficient Markets Hypothesis

This says that the best way to judge the expected return is to look at the market price. There are three forms

Weak — Market prices incorporate all information from price history. If this holds, chartism is no better than random picking

Semi strong — Market prices incorporate all public information. This definition is unclear because public is not well defined. However, it would be reasonable to believe that markets with higher disclosure levels are more efficient.

Strong — Market prices incorporate all information. If this holds, laws on insider trading are useless.

The question of market efficiency has a high bearing on investment management. In efficient markets, active fund managers are a drain on resources. However, it is the active managers that make them efficient in the first place.

There have been numerous studies which have tried to prove or disprove EMH. Attempts to disprove it included noting overreaction (to past performance and growth) and underreaction (to mergers and demergers, and earnings announcements). An approach by Shiller was to claim stock market prices were excessively volatile. A lot of the studies have been flawed in their assumptions, use of statistics and handling of the problems of risk.

Measures of Risk

The next sections will examine the relationship between return and risk. Return is almost always defined as the expected investment return. Risk can be

Variance of return $\int_{-\infty}^{\infty} (\mu - x)^2 f(x)\, dx$

The most popular due to mathematical tractability

Corresponds to a quadratic utility function

Semivariance of return $\int_{-\infty}^{\mu} (\mu - x)^2 f(x)\, dx$

Proportional to variance for symmetrically distributed assets.

Corresponds to a utility function quadratic below and linear above μ.

Expected shortfall $\int_{-\infty}^{L} (L - x) f(x)\, dx$

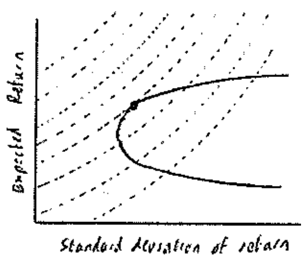
Corresponds to a utility function with a discontinuity at L.

Mean Variance Portfolio Theory

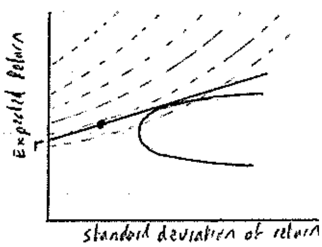
Assume an investor is non saitiable, risk adverse and selects investments only on the basis of mean and variance of returns over a fixed time scale.

A portfolio is inefficient if the investor can find another portfolio that gives higher returns for the same risk, or lower risk for the same returns.
A portfolio is efficient if the investor can find no such portfolio

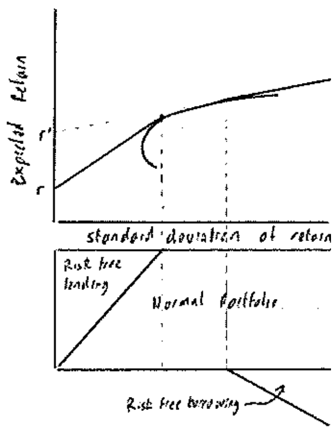
Suppose we have securities $1, \ldots, N$ with returns $R_1, \ldots, R_N$ expected returns $E_i = E(R$ covariances C_{ij} and variances $V_i = Var(R_i) = C_{ii}$. Let P be a portfolio investin a proportion x_i in each security. Then $E_P = x^T E$ and $V_P = x^T C x$.



Minimising V subject to a given E, we get a graph similar to that on the right. The investor chooses a portfolio on the efficient frontier where the frontier is at a tangent to an indifference curve (shown dashed)



In many circumstances, investors also have the option of investing in a risk free security. This has the effect of making the efficient frontier a straight line.



The shape of the curve also changes if borrowing / short selling is not allowed, or if the rate of borrowing is higher than the risk free rate. This gives rise to corner portfolios, which occur when one security reaches its upper or lower limit. This can be seen by a kink in the curve.

Finding V for a given E is easy when there are only two securities. Simply find the proportions x_1 and x_2 that give E and substitute them into the formula for V. Differentiating this formula also tells you that the global minimum for V occurs when $x_1 = \frac{V_2 - C_{12}}{V_1 + V_2 - 2C_{12}}$.

With more securities we use the lagrangian $W = x^T C x - \lambda(x^T E - E_p) - \mu(x^T I - 1)$ to minimise $V = x^T C x$ subject to $x^T E = E_p$ and $x^T I = 1$. Differentiating W with respect to all x_i's, λ and μ gives the linear equations, which we can solve.

$$\begin{pmatrix} 2C & -E & -I \\ E^T & 0 & \\ I^T & & \end{pmatrix}\begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ E_p \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 2C & -E & -I \\ E^T & 0 & \\ I^T & & \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ E_p \\ 1 \end{pmatrix}$$

Even without solving, V is quadratic and x_i are linear in E.

If borrowing / short selling is not allowed, we would have to check the frontier calculated to ensure all parts of it meet the constraints, remove the bit which did not, and recalculate. If additional constraints were added, we would have to add terms to W (such as $v_1(x_1 + x_2 - b)$), which may incorporate slack variables z_i if the constraints are inequalities. This is complicated, so it is unlikely to get asked.

Capital Asset Pricing Model

This extends mean variance portfolio theory to cover the entire market. Assume all investors are non saitiable, risk adverse and select investments only on the basis of mean and variance of returns over a fixed time scale. In addition assume

- Investors have the same time horizon
- Investors have the same value of money (no currency issues, inflation issues)
- Perfect markets - information is freely available, no investor can affect security prices
- Investors have the same estimates of mean, variance and covariance of securities
- All investors can borrow or lend unlimited amounts at the risk free rate.

We immediately get the result that all investors have the same efficient frontier, and thus should choose a portfolio of the form a% risk free + $(100-a)$% efficient risky portfolio. The fact that the optimal combination of risky assets can be found without knowing an investors preferences towards risk and return is called the separation theory.

The line of the efficient frontier is called the capital market line and is given by the equation $(E_p - r)/\sigma_p = (E_m - r)/\sigma_m$, where E_m, σ_m are the return and risk of the market portfolio, and E_p, σ_p are those for an efficient portfolio.

The CAPM has rather unrealistic assumptions and empirical studies do not give strong support. Testing is also hard, because assets include everything one could invest in (human capital), and the expected return on these is near impossible to measure. Nonetheless, studies do provide some evidence of a linear relationship between risk and return, so the main thrust probably holds.

Beta Values

This result holds for both MVPT and CAPM. Given a portfolio Q and an efficient portfolio Q, the expected return on Q is given by

$$E_Q - r = \beta_{Q,P}(E_P - r) \qquad \text{where } \beta_{Q,P} = \mathrm{cov}[R_Q, R_P]/V_P$$

We find this by taking a portfolio with proportion x of P and $1-x$ of Q. Then

$$\left.dE/dx \,/\, d\sigma/dx\right|_{x=1} = \left.dE/d\sigma\right|_{x=1} = \frac{(E_Q - E_P)\sigma_P}{C_{Q,P} - V_P}$$

Equating this to the tangent of the efficient frontier $\frac{E_P - r}{\sigma}$ gives the result.

In CAPM, the line $E_Q - r = \beta_Q(E_M - r)$ is called the security market line, and allows the expected return on any portfolio to be evaluated.

Multifactor Models

We have examined the relationship between risk and return. We will now loo at models of future returns. The most general is the multifactor model.

$$R_i = a_i + b_{i,1} I_1 + \dots + b_{i,L} I_L + c_i$$

The values I_j are a set of L factors which the returns depend on; a_i, $b_{i,j}$ show the extent of the (linear) dependance and c_i is an error term.

There are three types of models, depending on the factors used

Macroeconomic This uses only macroeconomic factors such as inflation, grow short term interest, etc. A related class also uses industry specific factors.

Fundamental This type also uses company specific factors such as beta, level of gearing, price earnings ratio, level of R&D, industry, etc.

Statistical Principle components analysis gives the most significant fact These typically have no economic meaning, and vary between data sets.

When constructing the model, we want significant uncorrelated factors.
We get the former by seeing the effect on the variance of c_i of removing factors
We get the latter by taking appropriate linear combinations of original factors.

The single index model is a simplified model tying in with CAPM

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

It has the following $E_i = \alpha_i + \beta_i E_M$

$V_i = \beta_i^2 V_M + V_{\varepsilon_i}$ [Systematic and specific risk]

$C_{ij} = \beta_i \beta_j V_M$ [No need to estimate all C_{ij} separately

We estimate α and β using time series regression analysis.

Lognormal Model

This models security prices with $\log S_u - \log S_t \sim N[\mu(u-t), \sigma^2(u-t)]$
Rewriting as $S_u = S_t \exp(X_{u-t})$ where $X_{u-t} \sim N(\mu(u-t), \sigma^2(u-t))$ it is clea
that S_u is lognormally distributed. All standard properties follow.

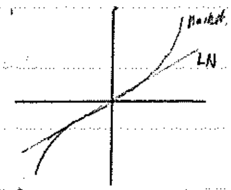
We will discuss the failings of this model in some detail.

volatility — Our assumption that σ is constant is not born out in reality - estimates vary wildly depending on the time period considered and sampling frequency. One solution is to model volatility separately.

drift — There are good theoretical reasons to suppose μ varies with time, which it does not do in this model.

mean reversion — One unsettled question is whether markets are mean reverting. There is evidence both for this being the case, and momentum effects (the opposite).

normality — Market crashes and days of no change occur more often than the normal distribution suggests. However by choosing a distribution that fits the market better, we loose the tractability that normal and lognormal distributions give.



When investigating time series such as this one, there are two ways of doing so

cross sectional properties. Fix the time and run the simulation lots of times, and look at the distribution formed.

longitudinal properties — Fix the simulation and look at all the time values and look at the values taken. This may identify limiting distributions.

In pure random walk environments, asset returns are independant over years, so we can equate both properties. Lack of equality in samples suggests this model is incorrect.

Arbitrage Pricing Theory

Let stocks have returns linearly related to indices as in the multifactor model.

$$R_i = a_i + b_{i,1} I_1 + \dots + b_{i,L} I_L + c_i$$

Then all securities and portfolios have expected returns described by the hyperplane

$$E[R_i] = \lambda_0 + \lambda_1 b_{i,1} + \dots + \lambda_L b_{i,L}$$

with appropriate values of $\lambda_0 \dots \lambda_L$.

It is called arbitrage pricing theory because any return not on this plane would give an arbitrage opportunity, and people exploiting this would force the anomalous return and the plane together again.

Efficient Market Hypothesis

This says that the best way to judge the expected return is to look at the market price. There are three forms

Weak Market prices incorporate all information from price history
If this holds, chartism is no better than random picking.

Semistrong Market prices incorporate all public information
This is unclear as there is no agreed definition of public. However, it would be reasonable to belive markets with a higher disclosure levels are more efficient

Strong Market prices incorporate all information
If this holds, laws on insider trading are useless.

The question of market efficiency has a high bearing on investment management. In efficient markets, active fund managers are a drain on resources. However, it is the active managers who make them efficient in the first place.

There have been studies which both prove and disprove EMH conclusively. Both are usually flawed in their assumptions, use of statistics, and ignoring problems of risk.

The Wilkie Model

A general ARMA (p,q) process is defined by

$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \dots + \alpha_p (X_{t-p} - \mu) + e_t + \beta_1 e_{t-1} + \dots + \beta_q e_{t-q}$$

It is discrete time and for appropriate α_i it is mean reverting. It forms the basis for a number of models including the Wilkie model.

It is important to understand the notation. Each variable has a main part

- Q Value of the consumer price index
- I Force of inflation
- Y Equity dividend yield at the end of the year (+ve)
- D Dividend income
- K Force of dividend growth during the year
- R Real yield on perpetual index linked bonds at the end of the year (+ve)

and sometimes a suffix. The following have special meaning

- MU Mean
- A Autoregressive parameter
- M Exponential moving average of previous values
- N Part of variable with zero mean

- SD Standard deviation
- Z $N(0,1)$
- E Residule - equivalent to XSD.XZ

By abuse of notation, we sometimes consider DM and YN to be variables in their own right.

The Wilkie model has several equations building on each other

$$I(t) = QMU + QA\,(I(t-1) - QMU) + QE(t)$$

$$YN(t) = YA.\,YN(t-1) + YE(t)$$
$$Y(t) = YMU\,\exp(YW.\,I(t) + YN(t))$$

$$DM(t) = DD.\,I(t) + (1-DD)\,DM(t-1)$$
$$DI(t) = DW.\,DM(t) + DX\,I(t)$$
$$K(t) = DI(t) + DMU + DY.\,YE(t-1) + DB.\,DE(t-1) + DE(t)$$
$$D(t) = D(t-1)\,\exp(K(t))$$

$$R(t) = RMU\,\exp(RA(\ln R(t-1) - \ln(RMU)) + RBC.\,CE(t) + RE(t))$$

They are horribly complex, but fortunately any questions asked are likely to give you the relevant bits of equation, so memorisation is not required.

With a bit of rearrangement, they can be put into vector form, and by adding some additional variables make it a markovian model, of the form

$$U(t+1) \mid U(t) \sim N(A_1.\,U(t),\; V_1 = LL^T)$$

Repeatedly projecting forward, we get

$$U(t+h) \mid U(t) \sim N(A_h\,U(t),\; V_h) \quad \text{with} \quad \begin{aligned} A_{h+1} &= A_1 A_h \\ V_{h+1} &= V_1 + A_1 V_h A_1^T \end{aligned}$$

Analysing this, we find longitudinal volatilities higher than crossectional ones. This is unsurprising because the crossectional ones have dependance on starting conditions. Dependance on term is also visible. In comparison real dividends are close to a random walk.

Total Return

The total return over a year for an equity is given by

$$\frac{P(t+1) + D(t+1)}{P(t)} = \frac{Y(t)\ (1 + 1/Y(t+1))}{D(t)\ /\ D(t+1)}$$

Similarly, the total return on an index linked bond is

$$\frac{P(t+1) + D(t+1)}{P(t)} = \frac{R(t)\ (1 + 1/R(t+1))}{Q(t)\ /\ Q(t+1)}$$

We can define the equity risk premium as the conditional expectation of the log relative return on equities and index linked gilts

$$E\left(\ln \left(\frac{Y(t)\ (1 + 1/Y(t+1))\ \exp(K(t+1))}{R(t)\ (1 + 1/R(t+1))\ \exp(I(t+1))} \right) \Bigg|\ U(t) \right)$$

$$= \log(Y(t)/R(t)) + E\left(\log(1 + 1/Y(t+1)) - \log(1 + 1/R(t+1)) + K(t+1) - I(t+1)\ |\ U(t\right.$$

Finding Models

Estimating parameters is a time consuming aspect of stochastic asset modelling.

In particular, there is great difficulty in interpreting data that appears to invalidate the model. One course of approach is to ignore it or to add in fudge factors to account for it. These actions could lead to oversimplified models which reflet more what the modeller hoped would happen than reality.

Another approach is to reject the model and try a more general one. The result here threatens to be a nontradable model of little use for its purpose. In addition, it leads to models that wrap round the data - that is, they become good at simulating the data, not economic reality.

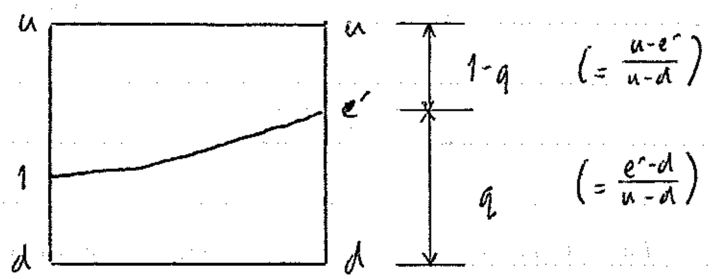
The trouble with stochastic models is that they create situations such as arbitrage opportunities that would not occur in real markets. For example, yields fluctuate more than they would if markets were efficient.

We can mitigate the problem by recalibrating models to give appropriate answers - a crude example of an equilibrium approach

An alternative is to use economic theory to achieve a statistically adaquete model which can also be rationalised in an efficient market framework. This also gives a concrete way of interpreting model output. However, use of the models to support decisions is problematic since they reflect assumptions put into them.

The Binomial Model

Suppose we have a stock which may go up by u or down by d, risk free cash with a return of e^r, and a derivative with value c_u if the stock goes up and c_d if the stock goes down.



derivative returns c_u

derivative returns c_d

Then the value of the derivative is

$$V_0 = e^{-r}\left(q c_u + (1-q) c_d\right)$$

We can find this by holding a replicating portfolio of ϕ stock and ψ cash

$$V_1 = \begin{cases} c_u = \phi u + \psi e^r & \text{(if the stock goes up)} \\ c_d = \phi d + \psi e^r & \text{(if the stock goes down)} \end{cases}$$

and solving the equations.

We can get more complicated examples, but they follow the basic principles.

Black Scholes

There is a method that can prove both the black scholes equation and the binomial model to be correct.

1. Establish a measure so the discounted asset price is a martingale

$$D_t = e^{-rt} S_t \qquad\qquad \tilde{D}_t = e^{-rt} \tilde{S}_t$$

2. Propose a fair price for the derivative at time t. (X is the payment at time T)

$$V_t = e^{-r(T-t)} E_Q [X \mid F_t] \qquad\qquad V_t = e^{-r(T-t)} E_Q [X \mid F_t]$$

3. Set $E_t = e^{-rt} V_t$. This is a martingale under Q.

4. By the martingale representation theorem, there is a process such that

$$dE_t = \phi \, dD_t \qquad\qquad dE_t = \tilde{\phi}_t \, d\tilde{D}_t$$

5. If we hold ϕ units of stock and $\psi = E_t - \phi D_t$ units of cash, then we hold a portfolio with value V_t. The change in V_t over a short period of time is $dV_t = d(e^{rt} E_t) = e^{rt} dE_t + re^{rt} E_t \, dt$.
But this equals the investment gain $\tilde{\phi}_t dS_t + \psi_t d(e^{rt}) = e^{rt} \phi \, dD_t + re^{rt}(\phi D + \psi) dt$
So the portfolio is self-financing, with $V_T = X$, and thus a replicating portfolio.

Black Scholes

A more direct method of solving this is to consider a portfolio of -1 derivati worth $-f(t, S_t)$, and $\partial f(t,S_t)/\partial S_t$ shares. This has value

$$V(t, S_t) = -f(t, S_t) + \partial f(t,S_t)/\partial S_t \; S_t$$

Assuming no net investment, the change of value in time dt is.

$$\begin{aligned} dV(t, S_t) &= -\left(\partial f/\partial t \; dt + \partial f/\partial S_t \; dS_t + \tfrac{1}{2} \partial f/\partial S_t^2 (dS_t)^2\right) + \partial f(t,S_t)/\partial S_t \left(dS_t + q S_t dt\right. \\ &= \left(-\partial f/\partial t + q S_t \partial f/\partial S_t - \tfrac{1}{2} \partial^2 f/\partial S_t^2 \sigma^2 S_t^2\right) dt \end{aligned}$$

As this is purely a function of dt as opposed to dS_t, the change is risk free. Thus it must match the rate of a risk-free bond

$$dV(t, S_t) = r V(t, S_t) \, dt$$

Substitute this in to give

$$\partial f/\partial t + (r - q) S_t \; \partial f/\partial S_t + \tfrac{1}{2} \sigma^2 S_t^2 \; \partial^2 f/\partial S_t^2 = rf$$

Apply boundary conditions and solve.

Results of Black Scholes

The end result is that the Garman-Kohlhagen formula satisfies black scholes. To show this works, we firstly calculate $\partial d/\partial t$, $\partial d/\partial s$ and $\partial^2 d/\partial s^2$, use these to calculate $\partial f/\partial t$, $\partial f/\partial s$ and $\partial^2 f/\partial s^2$ [we need the identity as follows]

$$\phi(d_1)/\phi(d_2) = e^{-\frac{1}{2}d_1^2 + \frac{1}{2}(d_1 - \sigma\sqrt{u})^2} = \frac{K}{S} e^{-(r-q)u}$$

and then substitute these into the formula.

We need to know the following notation

K	option price of share
S	share price
u	time period
r	risk free rate of return
q	dividend return
σ	volatility of share price.

and what the following symbols represent

θ	$= \partial f/\partial t$	Rate of growth of the portfolio
Δ	$= \partial f/\partial s$	Delta hedging gives us the matching portfolio. High stock if in-the-money.
Γ	$= \partial^2 f/\partial s^2$	Large gamma gives higher rebalancing costs. High if close to balance point.
ρ	$= \partial f/\partial r$	
λ	$= \partial f/\partial q$	
K	$= \partial f/\partial \sigma$	Sensitivity to changes in volatility.

Black scholes assumes a complete, perfect market, with infinitely divisible assets, unlimited short-selling, continuous trading. It assumes shares follow a brownian process, there are no arbitrage opportunities and the risk free rate of interest is constant