Mortality for Multiple Lives

In single life functions, we can view x as a status - 'the life is alive'. When the status becomes false, we pay the assurance or stop paying the annuity. We deal with multiple lives by adding statuses.

xy	Both x and y are alive
$\overline{xy}$	One of x or y is alive.
$\overset{1}{x}y$	Both x and y are alive. The status changes only if x dies first.
$\overset{2}{x}y$	One of x or y is alive. The status changes only if x dies second.
$\overset{1}{\overbrace{xy}}:\overline{n}\rceil$	Both x and y are alive and fewer than n years have passed

A number of results are immediate from the definitions.

$$A_{xy} + A_{\overline{xy}} = A_x + A_y$$

$$A_{xy} = 1 - d\,\ddot{a}_{xy}$$
$$\bar{A}_{xy} = 1 - \delta\,\bar{a}_{xy}$$

$$\ddot{a}_{xy} + \ddot{a}_{xy} = \ddot{a}_x + \ddot{a}_y$$

$$A_x = A_{\overset{1}{x}y} + A_{\overset{2}{x}y}$$
$$A_{xy} = A_{\overset{1}{x}y} + A_{x\overset{1}{y}}$$
$$A_{\overline{xy}} = A_{\overset{2}{x}y} + A_{x\overset{2}{y}}$$

Annuities using $\overset{1}{x}y$ and $\overset{2}{x}y$ are not commonly used

If we assume the lives are independant, many more results follow

$${}_tp_{xy} = {}_tp_x\,{}_tp_y \qquad \left[\Rightarrow \frac{l_{x+t:y+t}}{l_{xy}} = \frac{l_{x+t}}{l_x}\frac{l_{y+t}}{l_y} \quad \text{so} \quad l_{xy} = l_x l_y \right]$$

$${}_tq_{\overline{xy}} = {}_tq_x\,{}_tq_y$$

$$\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t} \qquad \Big[\text{as } \mu_{x+t:y+t} = \frac{-1}{l_{xy}} \frac{d}{dx} l_{x+t:y+t}$$

Most other relationships are built up around the known future lifetime.

$$T_{xy} = \min\{T_x, T_y\} \qquad T_{\overline{xy}} = \max\{T_x, T_y\}$$

$$P(K_{xy} = k) = {}_{k|}q_{xy} \qquad P(K_{\overline{xy}} = k) = {}_{k|}q_{\overline{xy}}$$

$$F_{T_{xy}}(t) = P[T_{xy} \leq t] = 1 - {}_tp_{xy} \qquad F_{T_{\overline{xy}}}(t) = (1 - {}_tp_x)(1 - {}_tp_y)$$

$$f_{T_{xy}}(t) = \tfrac{d}{dt}(1 - {}_tp_{xy}) = {}_tp_{xy}\,\mu_{x+t:y+t} \qquad f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$$

These can be used to create integrals as for the single life case.

$$\bar{A}_{xy} = E(\bar{A}_{\overline{T_{xy}|}}) = \int v^t\,{}_tp_{xy}\,\mu_{x+t:y+t}\,dt \qquad Var(\bar{A}_{\overline{T_{xy}|}}) = {}^2\bar{A}_{xy} - \bar{A}_{xy}^2$$

$$A_{xy} = E(A_{\overline{T_{xy}|}}) = \sum v^{t+1}\,{}_tp_{xy}\,q_{xy} \qquad Var(A_{T_{xy}}) = {}^2A_{xy} - A_{xy}^2$$

$$\bar{A}_{xy} = E(\bar{A}_{\overline{T^1_{xy}|}}) = \int v^t\,{}_tp_{xy}\,\mu_{x+t}\,dt \qquad Var(\bar{A}_{T^1_{xy}}) = {}^2\bar{A}^1_{xy} - (\bar{A}^1_{xy})^2$$

$$A^1_{xy} = E(A_{\overline{T^1_{xy}|}}) = \sum v^{t+1}\,{}_tp_{xy}\,q^1_{xy} \qquad Var(A_{T_{1y}}) = {}^2A^1_{xy} - (A^1_{xy})^2$$

When calculating variance, ${}^2A$ means to calculate using interest of $i^* = 2i + i^2$
We will also need to consider reversionary annuities.

$$a_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

$$a_{x|y} = \int v^t\,\bar{a}_{y+t}\,{}_tp_{xy}\,\mu_{x+t}\,dt$$

Finally, there are joint commutation functions

$$D_{xy} = v^{\frac{x+y}{2}}\,l_{xy} \qquad N_{xy} = \sum_{\omega-\max(x,y)}^{\omega-\max(x,y)} D_{x+t:y+t} \qquad \ddot{a}_{xy} = \frac{N_{xy}}{D_{xy}}$$

$$C_{xy} = v^{\frac{x+y}{2}}\,d_{xy} \qquad M_{xy} = \sum C_{x+t:y+t} \qquad A_{xy} = \frac{M_{xy}}{D_{xy}}$$

Multiple Decrement Tables

Single decrement tables only consider a member leaving one way (e.g. death). Multiple decrement tables allow multiple exits (e.g. dying, withdrawal, ill health, retirement). However, they are less tractable, so we must devise a method to switch between them.

First some notation. Consider a population $(al)_x$ of age x subject to decrements $1, \dots, n$. Let $(ad)_x^i$ be the number of lives lost to decrement i and $(aq)_x^i = (ad)_x^i / (al)_x$. Define $(aq)_x$ as the probability any decrement occurs, and $(aq)_x^{-i} = (aq)_x - (aq)_x^i$.

These decrements are dependant, because the value of each depends on all the others. For each decrement, we wish to construct a table only involving that decrement. These are independant decrements, and we write them as q_x^i.

Assume each decrement has independant transition intensities $(a\mu)_x^i = \mu_x^i$ and that the decrements are uniformly distributed across the year. During one year, $(ad)_x^i$ people are lost to non i decrements. If they were not lost each would have an extra half year to be taken by the i decrement. Thus

$$q_x^i = (aq)_x^i + \tfrac{1}{2} q_x^i (aq)_x^{-i} \quad \Rightarrow \quad q_x^i = (aq)_x^i \,/\, 1 - \tfrac{1}{2}(aq)_x^{-i}$$

Converting back is similar. The dependant decrement is equal to the independant decrement, minus half a year for lives lost to other decrements, plus a third for those doublecounted, etc.

$$(aq)_x^i = q_x^i \left(1 - \tfrac{1}{2} q_x^{j \neq i} + \tfrac{1}{3} q_x^{j \neq i} q_x^{k \neq j, k \neq i} - \tfrac{1}{4} q_x^{j \neq i} q_x^{k \neq j, k \neq i} q_x^{l \neq k, l \neq j, l \neq i} + \dots \right)$$

Variations in Mortality

In addition to age and sex, mortality rates vary over time and by location and class. However, these are not causal - they act as proxies for the proximate determinants.

Nutrition	Bad nutrition, lack of the right vitamins and minerals, and overeating can lead to sickness and poorer recovery in various ways.
Occupation	This determines a person's environment for 40 hours or so each week including level of danger, exercise and stress. It also influences lifestyle through the level of income provided. Some jobs also attract healthy workers or have health checks.
Housing	This includes the physical quality of a house and how it is used. Overcrowding has a particular effect on infectious diseases.
Geographic	On a global scale, this is called climate. Areas differ in their level of disease, natural disasters, political unrest, food production, etc. We also consider access to facilities such as hospitals.
Educational	This encompasses better income, choosing a better diet and more exercise, personal health care, moderation in alcohol and smoking, awareness of problems with drugs and sex, etc.

Actuarial work involves pooling similar risks together. Recognising where groups are not similar, and splitting them into more homogeneous groups can improve the accuracy of the answers. This process is called selection - splitting it by the classes above is class selection, splitting it by time is time selection. If we select by non-causal elements it is called spurious selection.

Single Figure Indices

These are weighted averages of functions of age specific rates, designed to allow easy comparison between homogeneous groups, or the same group over time. Their main advantage is their simplicity for summary and comparison. However, they require extensive data, loose a lot of information, and have distortions due to the weightings. The methods use the following notation

$E^c_{x,t}$ Central exposed to risk (total life-years at risk) in population aged x to $x+t$.
$m^c_{x,t}$ Central rate of mortality in population aged x to $x+t$
${}^sE^c_{x,t}$ Central exposed to risk for a standard population
${}^sm^c_{x,t}$ Central rate of mortality for a standard population

Crude mortality rate $\left(\frac{\sum {}^sE^c_{x,t}\, m^c_{x,t}}{\sum E^c_{x,t}}\right)$

This measures actual deaths per exposed to risk. Its main problem is that populations with different age structures get different weightings and so cannot reasonably be compared.

Standardised mortality ratio $\left(\frac{\sum E^c_{x,t}\ m^c_{x,t}}{\sum E^c_{x,t}\ {}^sm^c_{x,t}}\right)$

This is the actual over expected deaths in the population

Directly standardised mortality rate $\left(\frac{\sum {}^sE^c_{x,t}\, m^c_{x,t}}{\sum {}^sE^c_{x,t}}\right)$

This is a weighted average of $m_{x,t}$ using ${}^sE^c_{x,t}$ as weights

Indirectly standardised mortality rate $\left(\frac{\sum {}^sE^c_{x,t}\ {}^sm^c_{x,t}}{\sum {}^sE^c_{x,t}}\ \frac{\sum E^c_{x,t}\ {}^sm^c_{x,t}}{\sum E^c_{x,t}\ m^c_{x,t}}\right)$

This is the crude mortality rate for the standard population, times the standardised mortality ratio. It is a good approximation of the directly standardised mortality rate.

Select Tables

Two effects cause a group drawn from a population to differ from that population

Adverse Selection Certain lives are more interested in being part of the group than others. The group membership will be skewed towards such lives.

Temporary Initial Selection This is where a group is defined by a particular event with an effect on mortality which lessens after time. For example, going through the medical when buying life assurance gives temporarily better mortality.

Usually, we just have to mitigate these effects with the benefit of experience. However, temporary initial selection in life assurance always happens, so they have a tool to deal with it - select tables with rates for this better mortality.

When representing a select rate, we replace the x representing age with $[x]+y$, representing nearest age at selection plus complete years since then. All the standard formulas hold. After the temporary initial selection, rates are sufficiently similar to standard mortality that we revert to using that.

To create select tables, we record transitions during the period $\theta_{[x],r}$ and take censuses $P_{[x],r}(t)$ at various times t. Then central exposed to risk is

$$E^c_{[x],r} = \int^{t=T} P_{[x],r}(t)\,dt \approx \tfrac{1}{2} P_{[x],r}(0) + \sum_{t=1}^{T-1} P_{[x],r}(t) + \tfrac{1}{2} P_{[x],r}(T)$$

We must take care to adjust $P_{[x],r}(t)$ in cases that dates do not align. Once we have these, we can estimate transition rates and probabilities

$$\mathring{\mu}_{[x],r} = \frac{\theta_{[x],r}}{E^c_{[x],r}} \qquad \mathring{q}_{[x],r} = \frac{\theta_{[x],r}}{E^c_{[x],r} + \tfrac{1}{2}\theta_{[x],r}}$$

Choosing an appropriate start age k and radix l_k, we can now construct a select mortality table, using

$$q_{[x]+t} = 1 - \frac{l_{[x]+t+1}}{l_{[x]+t}}$$

Population Projection

This is useful for estimating the demand for goods and services, what populations could have been in the past, etc. Simple projections can be expressed in the form

$$\frac{1}{P(t)} \, d/dt \, P(t) = \rho - k\, f(P(t))$$

In the two simplest cases, $f(x) = 0$ and $f(x) = x$. Slightly more complicated possibiliti include the ARIMA time series. These models do not tend to be sucessful.

The compound method projects a population subdivided in age groups by looking at the factors that cause it to change - fertility, migration and mortality.

$$P_x(n) = P_{x-1}(n-1)\cdot(1 - q_{x-\frac{1}{2}}(n-1)) + M_x(n)$$
$$P_0(n) = B(n)\cdot(1 - \tfrac{1}{2} q_0(n-1)) + M_0(n)$$

Births and migrations are modelled separately. For example births may be modelled b

$$B(n) = \sum \tfrac{1}{2} f_x(n)\,(P^f_x(n-1) + P^f_x(n))$$

If we model males and females separately, we split the births by the appropriate ratio

Fertility rates are much more variable than mortality rates because they are influenced by many social and economic factors, such as availiability of contraception, economic and political prospects., percieved costs of raising a child and cultural influence, such as religion, labour force participation and social norms in general.

We typically estimate the total fertility rate for a group born at the same time, subtract the babies that they have already had, and allocate the remaining birth rate sensibly.

Gross Premium Reserving

This extends net premium reserving by incorporating expenses. These include

Overhead These do not vary with the amount of business written, and are usually allocat over all the policies in force that year as a renewal expense.

Direct These depend on the amount of business being written. They exclude commission, as this can be varied at will by the office. They are divided into

Initial (I) Arising when the policy is issued
Renewal (e) Arising regularly during the policy term
Termination (t) Arising when the policy terminates as a result of a claim

The gross future loss is now $Sv^{T_x} + I + e\bar{a}_{\overline{T_x}|} + fv^{T_x} - G\bar{a}_{\overline{T_x}|}$ which gives an equatio of value $S\bar{A}_x + I + e\bar{a}_x + f\bar{A}_x = G\bar{a}_x$. We sometimes also add a loading for profit. The prospective reserve is now $S\bar{A}_{x+t} + e\bar{a}_{x+t} + f\bar{A}_{x+t} - G\bar{A}_{x+t}$, and the retrospective reserve is $D_x/D_{x+t}\,(G\bar{a}_{x+t} - S^1_{x:\overline{t}|} - I - e\bar{a}_{x:\overline{t}|} - f\bar{A}^1_{x:\overline{t}|})$. If both are calculated on the same basis, they are the same.

It is clear that gross and net reserves are linked. Indeed, if there are no termination expenses, the difference between the reserves at time t is $-I\frac{\bar{a}_{x+t}}{\bar{a}_x}$.
This is called the Zillmer adjustment, and compensates for the amount that net premium reserves were over-reserving, due to the company recieving gross rather than net premium We find the value by comparing prospective reserves under gross and net methods.

As with net premiums, we can roll the gross premiums forward

$$({}_tV + G - e)(1+i) - q_{x+t}(S - {}_{t+1}V' + f) = {}_{t+1}V'$$

With Profits

With profits funds allow life companies to vary the premiums and/or benefits to pass on emerging surplus following a valuation. This allows them to make slightly pessimistic assumptions about interest, mortality, etc which protects against loss, while offering the policyholder more than they would otherwise have recieved if things go well. With profits are suitable for endowment assurances, whole life, deferred and immediate annui

Bonuses usually follow the annual valuation and are added according to one of these methods

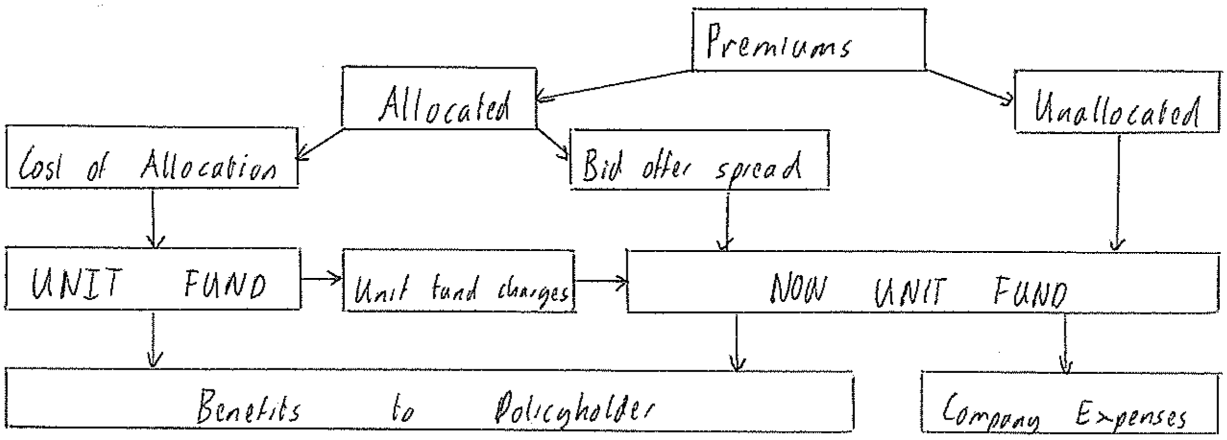
Simple	A proportion of the original sum
Compound	A proportion of the original sum and bonuses so far
Super compound	A proportion of the original sum and a higher proportion of bonuses so far
Terminal	Any bonus added at the end.

The asset share is widely used for calculating bonuses. It consists of premiums and a proportion of non-profits profits, less deductions, accumulated at the actual rate of return. Deductions include cost of providing benefits (guarantees, insurance), direct expenses, transfer of profits to shareholders, tax on investment income, commission, cost of capital for early years an contribution to free assets. Asset shares can be for individual or groups of policies.

How the bonus is calculated depends on the policy. Typically, we smooth the asset value, look at last times reversionary bonus, grant a similar one, and use the termin bonus as a balancing item.

Unit Linked Policies

A specified proportion of each premium is used to buy units in an investment vehicle
On payout, the units are used to buy benefits. The cashflows are complex



Cashflows include premiums and expenses at various times, other cash benefits and recei
direct and indirect benefits to be paid and what reserves are required. The balancing item is the profit that the company makes.

Plotting cashflows over a period of years we get a number of profits - the profit vecto
The profit signature is the vector of expected profits per policy. When comparing policies the profit signature is too complex, so we use one of the following.

Net Present Value $\quad \Sigma (1+i_d)^{-t} (PS)_t$

Profit Margin $\quad \Sigma (1+i_d)^{-t} (PS)_t \, / \, \Sigma (1+i_d)^{-(t+1)} \, {}_{t-1}p_x \, P_t$

We often specify the profit criterion to be one of the above equalling a proportion of commission, so that if the salesforce maximises its income, profits will be maximised

Reserving using Cash Flows

A reserve is a sum of money put aside to meet future liabilities. It must be sufficient even when actual experience turns out to be worse than expected. Thus an actuary must determine a prudent reserve basis. Note that the additional reserving will affect the pricing cashflows, so the process is an iterative one.

After we have calculated the profit signature on a prudent basis, we must zeroise it. This means setting all negative cashflows to zero by reducing earlier positive cashflows, or failing that by reducing the initial negative cashflow, using the following

$$X'_{m-1} = X_{m-1} + p_{x+m-2} \frac{X_m}{1+i}$$

This means, once sold and funded at the start, the policy is self supporting.

The initial costs and establishing reserves causes new business strain. Actuarial funding reduces this by taking money from future profits. Essentially, if we take the portio of the management charge that goes into profits at the end, and use this to buy units we could have bought fewer units at the start and used the saved money to pay initial cost

In order to get enough units at the end, at time t, the company must hold at least a proportion $A_{x+t:\overline{n-t|}}$ valued at $i = \frac{\text{charge for act funding}}{1 - \text{man charge}}$ of the units it should be holding. To be prudent, it should hold sufficient units to meet the policies surrender value.

Alterations to Contracts

There are two main ways in which policyholders can alter their contracts

Surrender Here they terminate it early in return for an immediate cash payment

Paid Up Here they stop paying premiums, but the value they have paid so far is used to fund reduced benefits

Other alterations are best dealt with on a one-off basis.

Both the retrospective policy value and the prospective policy value have a place in calculating the policies value. Typically it is a blend of the two, subject to a minimum of 0.

Guarantees

Unit linked policies often come with guarantees of minimum return. One method of pricing these is to look at the price of an option to sell the unit for the guarantee pric Where early surrender is possible an american option should be used. Buying this option can be used to mitigate the risk.

Another method is to set up a stochastic model and run a large number of simulations A premium can then be set which meets the value with high probability.

Another type of guarantee is an option to renew a contract on current terms. That is, the price is with select mortality but the value has ultimate mortality. Two methods are us

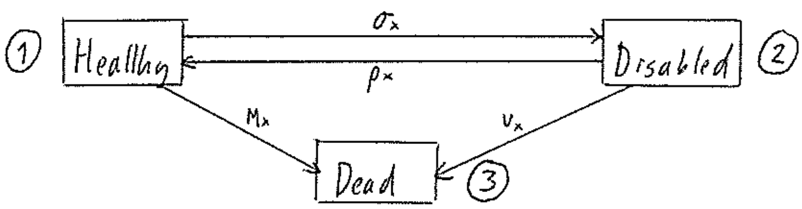
Conventional	Assume everyone takes the option. This is not possible with lots of dates
North American	Model people taking options as an extra decrement. This needs lots of data.

Disability

Permanent health insurance protects policyholders against loss of income during periods of illness in exchange for regular premiums at other times. Features may include

Waiting periods	Benefits are not paid for the first x weeks of the contract
Deferred period	Benefits are not paid for the first x weeks of sickness.
Off period	Short periods of health between illness are discounted.

There are three main ways benefits are calculated. The simplest is the multistate model



Given transition intensities σ, ρ, μ, ν, we calculate probabilities ${}_t p^{ab}_{x,z}$ of having been in state b for a period z at time x+t, given you were in state a at time x. Integrating these values gives the value of policies.

The second approach is to use decrement tables. One decrement table lists the rate of claims from sickness. The second shows rates of recovery (and death) for disabled people. We use the second to calculate an annuity for each period of disability, and the first to calculate the likelyhood of having to pay the annuity.
(Note that the annuities cover one period of sickness, and the first table includes recovered people)

The Manchester-Unity approach defines $\bar{z}_{x+t}$, the proportion of those alive at x+t who are sick. In order to calculate benefits varying by duration, these proportions are subdivided into $\bar{z}^{a/b}_{x+t}$ where the duration is between a and a+b. To find the initial and central rates of sickness (that is, over a year rather than instantaneously), we use

$$S_x^{a/b} = \frac{52.18 \int_0^1 l_{x+t}\, \bar{z}^{a/b}_{x+t}\, dt}{l_x} \qquad z_x^{a/b} = \frac{52.18 \int_0^1 l_{x+t}\, \bar{z}^{a/b}_{x+t}\, dt}{\int_0^1 l_{x+t}\, dt}$$

Pensions

Defined benefit pension schemes usually entitle members to a pension at normal retireme
age of a proportion of final pensionable salary for each year of service.
Additional benefits such as cash sums at retirement, entitlement to ill health early
retirement and death in service benefits may also be included. They are partially
funded by a fixed proportion of the members salary, with the balance paid by the compa

When a member leaves the scheme, they would be entitled to a return of contributi
a transfer value, a deferred pension or an immediate pension.

To value any of these, we value the cashflows from the pension in the usual wa
The main difference is consideration of salary increases, for which we use a decrement s
So salary averaged over three years preceding 65 would be $(s_{62}+s_{63}+s_{64})\, S/3s_{x-1}$