Pricing a Security

If someone is paying income tax t_1 and capital gains tax t_2,

- What price $A = P/N$ should they pay to get a yield i.
- What yield i should someone paying price A get.

No Capital Gains Tax

$$\begin{aligned} A(n,i) &= \text{present value of net interest payments} + \text{present value of gross capital payments} \\ &= D(1-t_1)\, a_{\overline{n}|}^{(p)} + C\left(\tfrac{1}{1+i}\right)^n \\ &= C + \left[(1-t_1)\,D/C - i^{(p)}\right] C a_{\overline{n}|}^{(p)} \end{aligned}$$

If the redemption date is optional, assume the borrower will choose the date minimising the return and thus choose the date minimising the price. That is

- $(1-t_1)D/C - i^{(p)} > 0 \Rightarrow$ earliest possible date
- $(1-t_1)D/C - i^{(p)} < 0 \Rightarrow$ latest possible date

To find the yield from the price, solve the equation of value for i. If the redemption date is optional, then

- $A > C \Rightarrow$ earliest possible date (minimise capital loss)
- $A < C \Rightarrow$ latest possible date (maximise capital gain)

Capital Gains tax

This is the same as the non-capital gains case, except the equations of value are

$$A(n,i) = \begin{cases} D(1-t_1)\, a_{\overline{n}|}^{(p)} + C\left(\frac{1}{1+i}\right)^n & (1-t_1)D/C - i^{(p)} \geq 0 \\ \left[D(1-t_1)\, a_{\overline{n}|}^{(p)} + (1-t_2)\, C\left(\frac{1}{1+i}\right)^n\right] \Big/ \left[1 - t_2\left(\frac{1}{1+i}\right)^n\right] & (1-t_1)D/C - i^{(p)} < 0 \end{cases}$$

When calculating the yield from the price, we know if capital gains tax is paid, and can simply write down an equation of value and solve it.

Deferred Income Tax

We simply change the equation of value to reflect this $A = D\, a_{\overline{n}|}^{(2)} - t_1 D v^k a_{\overline{n}|} + C v^n$

Inflation

Assuming inflation at rate e, we get adjusted cashflows $c_t^e = (1+e)^t c_t$.
This gives the real rate of interest $j(i,e) = \frac{i-e}{1+e}$. For e small, $j \approx i-e$.

For known inflation, we can adjust the equation of value

$$A = \sum D \frac{Q(k)}{Q(0)} \left(\frac{1}{1+i}\right)^k + C \frac{Q(n)}{Q(0)} \left(\frac{1}{1+i}\right)^n$$

For unknown inflation, we can still estimate it this way $[\, Q(1\tfrac{1}{2}) = Q(0)(1+j_1)(1+j_2)^{1/2}$

Calculating Value

When calculating value, we must consider two rules - eqⁿ of value and no arbitrage

- Present value of income − Present value of outgoings = 0
- It is impossible for an investor to make a deal giving nonzero probability of future profit with zero initial cost and no risk of future loss.

We will show how to use this by calculating the forward price of a security.

Security with no income

Let portfolio A contain a unit of a security bought for price S_0.
Let portfolio B contain a forward contract to buy that security at time T for price K, and contain an amount $Ke^{-\delta T}$ in a risk free investment with force of interest
At time T, the payout is the same, so the price of each must be the same.
Thus we get the result that $Ke^{-\delta T} = S_0$

Suppose at time r we sell either portfolio. Portfolio A will be worth the security.
Portfolio B will be worth the forward contract V_r and the risk free investment $Ke^{-\delta(T-}$
Thus $S_r = V_r + S_0 e^{\delta T} e^{-\delta(T-r)}$ and the value of the forward contract is $V_r = S_r - S_0 e$

Security with fixed income

Let portfolio A contain a unit of security bought for S_0, income put into risk free investmen
Let portfolio B contain a forward contract to buy that security at time T for price K an
contain $Ke^{-\delta T} + c_1 e^{-\delta t_1} + \ldots$ (where $c_1, t_1, \ldots$ correspond to income) in risk free investment.
This gives $K = S_0 e^{\delta T} - c_1 e^{\delta(T-t_1)} - \ldots$ and a value $V_r = S_r - S_0 e^{\delta r} - c_2 e^{\delta(r-t_2)}$

Security with known dividend yield

Let portfolio A contain e^{-DT} units of the security, dividends immediately reinvested.
Let portfolio B contain a forward contract to buy that security at time T for price K and contain $Ke^{-\delta T}$ in risk free investment. Now we get $K = S_0 e^{(\delta - D)T}$.

Immunisation

As interest rates rise, the values of assets and liabilities fall, and vice versa. Investors holding assets to pay off liabilities would want to immunise the fun so that interest rate changes do not cause the liabilities to exceed the assets. They can do this by ensuring the values obey Redington's conditions

- $V_A(i) = V_L(i)$

The present value $V = \sum C_{t_k} \left(\frac{1}{1+i}\right)^{t_k}$ of the assets equals that of the liabilitie

- $v_A(i) = v_L(i) \Rightarrow {dV_A(i)}/{di} = {dV_L(i)}/{di}$

The effective duration $v(i) = -\frac{1}{V}\frac{dV}{di}$ of the assets equals that of the liabilities.
We could use duration instead - $\tau = -\frac{1}{V}\frac{dV}{d\delta} = (1+i)\,v(i)$.
Duration shows the length of a zero coupon bond with the same volitility.

- $c_A(i) > c_L(i) \Rightarrow {d^2V_A(i)}/{di^2} > {d^2V_L(i)}/{di^2}$

The convexity $c(i) = \frac{1}{V}\frac{d^2V}{di^2}$ of the assets is greater than that of the liabilities.
A cashflow with spread payments is higher than one with bunched payments.
Convexity is a measure of the change in duration as interest rates change.

Derivatives

These are financial instruments to control risk, either reducing it (hedging) or increasing it (speculation) in order to enhance returns. There are three types

Futures

This is a contract to buy or sell a certain good at a certain price on a certain date. The buyer is the long party, the seller is the short party. Each party to a contract deposits a sum of money called 'margin' with the clearing house. Additional amounts are paid or withdrawn as the market moves. This allows the clearing house to minimise its exposure if one of the parties defaults on the contrac

Commodity futures are on assets such as sugar, cocoa, wheat, pigs and gold.

Bond futures allow the short party to deliver one of a range of government bond The price is adjusted to allow for the fact that the coupon rate may be different to the bond underlying the contract settlement price.

Stock index futures provide a notional transfer of assets underlying a stock index.

Currency futures require delivery of a set amount of currency.

Short interest rate futures have a value of 100 - 3 month interest rate on settlement

Options

These are similar to futures, except the holder of the option has the right to choose if the deal will go ahead. The writer of the option is paid a premium to compensate them for this. Only the writer pays margin.

A put option gives the holder the right to sell, a call option the right to buy. American style options may be exercised any day before expiry, European style only at expiry.

Swaps

These can be viewed as both sides exchanging securities with the same nominal amount and value but different payment types. There are two types

Interest rate (coupon swap)

One side pays fixed rate and the other side pays variable rate interest payments. This can reduce the overall cost of debt if each side has access to a cheap form of the appropriate type of borrowing.

Currency swaps.

The companies exchange lump sums in different currencies, pay interest in the currency they have borrowed, and swap the same size of lump sum back.

Stochastic Interest Rates

These tend to be hard to analyse directly due to the non-linear equations. For example, the accumulated value using the mean interest rate is different to the mean accumulated value.

Let S_n be the value of an investment of 1 at time 0 after n independent identically distributed interest rates with a yield which has mean j and variance s^2. While hard to analyse directly, finding moments is relatively easy.

$$E(S_n^k) = E\left(\prod_{t=1}^{n} (1+i_t)^k\right)$$
$$= \prod_{t=1}^{n} E\left((1+i_t)^k\right)$$

$$\Rightarrow \quad E(S_n) = (1+j)^n$$
$$E(S_n^2) = \left((1+j)^2+s^2\right)^n \Rightarrow V(S_n) = \left((1+j)^2+s^2\right)^n - (1+j)^{2n}$$

Let A_n be the value of an annual investment of 1 under the same conditions

$$A_n = (1+i_n)(1+A_{n-1})$$
$$E(A_n) = (1+j)(1+E(A_{n-1})) \quad \Rightarrow \quad E(A_n) = \ddot{s}_{\overline{n}|}^{\,j}$$

$$E(A_n^2) = (1+2i_n+i_n^2)(1+2A_{n-1}+A_{n-1}^2)$$
$$= \left((1+j)^2+s^2\right)\left(1+2E(A_{n-1})+E(A_{n-1}^2)\right)$$

The latter can only be found by following the recurrance relation.

There is one special case which is easy to analyse - suppose $1+i_t$ is log normally distributed - that is $\log(1+i_t) \sim N(\mu, \sigma^2)$. Now

$$S_n = \prod_{t=1}^{n} (1+i_t)$$
$$\log S_n = \sum_{t=1}^{n} \log(1+i_t)$$
$$\log S_n \sim N(n\mu, n\sigma^2)$$
$$S_n \sim \log N(n\mu, n\sigma^2)$$

Term Structure of Interest Rates

Let P_n be the price of an n year zero coupon bond yielding one pound.
Let $f_{t,n}$ be the annual yield of a bond starting at t, and $y_n = f_{0,n}$, $f_t = f_{t,1}$.
Let $F_{t,n}$ be the force of interest of a bond starting at t, and $Y_n = F_{0,n}$, $F_t = \lim_{n \to 0} F_{t,n}$

$$1 + y_n = \left(\frac{1}{P_n}\right)^{\frac{1}{n}}$$
$$1 + f_{t,n} = \left(\frac{P_t}{P_{t+n}}\right)^{\frac{1}{n}}$$
$$1 + f_t = \left(\frac{P_t}{P_{t+1}}\right)$$

$$f_{t,n} = e^{F_{t,n}} - 1$$

$$Y_n = \frac{1}{n} \log \frac{1}{P_n}$$
$$F_{t,n} = \frac{1}{n} \log \frac{P_t}{P_{t+n}}$$
$$P_t = e^{-\int_0^t F_s \, ds}$$

If we have a non-zero coupon bond, we define yield to maturity to be the internal rate of return and the par yield to be the coupon rate at which the price would be one pound for one pound nominal.

Interest rates change over time because of

- Supply and demand
- Change (or expected change) in base rates
- Interest rates in other countries
- Expected future inflation
- Tax rates

There are three main theories to explain the shape of spot rates against term.

- Market segmentation - supply and demand for terms gives the shape
- Liquidity preference - longer dated bonds are more sensitive to interest rate movements and so they need a higher yield to compensate
- Expectations theory - investors speculate on the change in interest rates - raising interest rates makes short term investments more attractive and lowering interest rates makes long term investments more attractive

Interest and Annuities

Simple Interest $\qquad C(1+ni)$

$C(1-nd)$

Compound Interest $\qquad C(1+i)^n$

$$1+i = v^{-1} = e^{\delta} = \frac{1}{1-d}$$

$$1+i = (1 + i^{(p)}/p)^p \qquad i^{(p)} = p\,((1+i)^{1/p} - 1$$

$$d^{(p)} = i^{(p)}$$

Annuities

$$a_{\overline{n|}} = \frac{1-v^n}{i} \qquad \ddot{a}_{\overline{n|}} = \frac{1-v^n}{d} \qquad \bar{a}_{\overline{n|}} = \frac{1-v^n}{\delta}$$

$$a^{(p)}_{\overline{n|}} = \frac{1-v^n}{i^{(p)}} \qquad \ddot{a}^{(p)}_{\overline{n|}} = \frac{1-v^n}{d^{(p)}}$$

$$s_{\overline{n|}} = \frac{(1+i)^n - 1}{i} \qquad \ddot{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{d} \qquad \bar{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{\delta}$$

$$s^{(p)}_{\overline{n|}} = \frac{(1+i)^n - 1}{i^{(p)}} \qquad \ddot{s}^{(p)}_{\overline{n|}} = \frac{(1+i)^n - 1}{d^{(p)}}$$

$$Ia_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} \qquad I\ddot{a}_{\overline{n|}} = 1 + a_{\overline{n|}} + Ia_{\overline{n-1|}} \qquad I\bar{a}_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{\delta}$$

$$\overline{Ia}_{\overline{n|}} = \frac{\bar{a}_{\overline{n|}} - nv^n}{\delta}$$

There are two ways of calculating the present capital value of a partially repaid loan. The prospective method is to sum the present value of all future payments. The retrospective method is to subtract the present value of all past payments from the present value of the loan.

The interest paid in a particular installment can be calculated by multiplying the capital value at the previous installment by i. The capital repaid by a particular installment can be found by either subtracting the interest from the payment or by subtracting the present capital value from the previous capital valu