Mathematical Tripos, Part III 1999

The Percolation Phase Transition

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I declare that this essay is work done as part of the Part III Examination. It is the result of my own work, and except where stated otherwise includes nothing which was performed in collaboration. No part of this essay has been submitted for a degree or any such qualification.

Signed . . . . Date . . . . .

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0. Contents

1. Introduction

There are many examples of physical systems which exhibit phase transitions, for example ice melting, or a ferromagnetic material becoming permanently magnetised. The percolation model is an example of a simple mathematical model which exhibits a phase transition and it is hoped that some insight may be gained by understanding the percolation model, which will transfer to the analysis of phase transitions in other physical systems. Equally, important ideas based on physical intuition such as *scaling theory*, and *the renormalisation group*, introduced originally with little or no mathematical rigour can be of great help in understanding these systems from a physical viewpoint.

The first part of this essay consists of a general overview of the percolation model near the critical point, and discusses the idea of *critical exponents* and their interrelations, without being particularly concerned with rigour. The second part presents a very brief account of what is proven in the two dimensional case, and the more major third part discusses the behaviour of the model in high dimensions, including a proof via the lace expansion of the existence of critical exponents in high dimensions. In many places results stated here are not 'best possible', for example many results have been restricted to bond percolation on the hypercubic lattice, since the technical details involved in the more general case seems mainly to hide the flow of the argument. In particular no discussion is given of the 'spread out lattice' on which many of the lace expansion results were originally proved. However the original paper of Hara and Slade (Hara and Slade, 1990a) contains a detailed discussion of this model.

1.1. The percolation model

There are two variants on *percolation* which will be considered in this essay, namely *site* and *bond* percolation. To start, take a lattice of some sort in d dimensional (Euclidean) space. This lattice will be denoted $\mathbb{L}^d$. The lattice consists of a set of vertices $\mathbb{V}$ (the *sites*) joined by a set of (possibly directed) edges $\mathbb{E}$ (the *bonds*).

Most of this essay will be concerned with the hypercubic lattice, which has vertex set $\mathbb{V}^d = \mathbb{Z}^d$, and edge set $\mathbb{E}^d = \{(x, y) : \|x - y\| = 1\}$. This pair $(\mathbb{V}^d, \mathbb{E}^d)$ will be denoted $\mathbb{L}^d$ as a short hand. Throughout d will universally be taken to denote the dimension of the space under consideration.

Site Percolation

In the site percolation model, each site is either open or closed. Each site is declared *open* with a probability p ($0 \leqslant p \leqslant 1$), independently of all the other sites. If a site is not open then it is closed.

Bond Percolation

The bond percolation model behaves similarly. Each bond is either open or closed, and it is declared open with a probability p independently of all the other bonds.

It turns out that all bond percolation problems can be described as site percolation problems on a 'covering' lattice of the original lattice. The converse is not true.

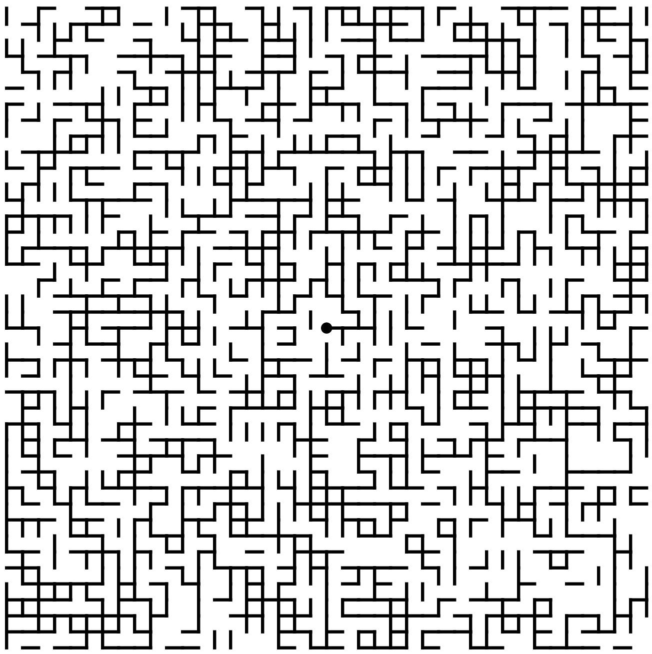


Figure 1. Simulation of Bond Percolation on a 40×40 fragment of the square lattice $\mathbb{L}^2$, at $p = p_c = 0.5$. The origin is represented by a black dot.

The Percolation Probability

Take a fixed site, which will be called the origin and denoted 0 (it is clear that the model is translation invariant in the choice of this site). Now consider the event that the origin is joined to infinitely many other sites by open bonds. Let the probability of this event, the *percolation probability* be denoted $\theta(p)$, where as before p is the probability of a given bond being open. As p increases from zero to one, for $d \geqslant 2$, $\theta(p)$ can be shown to be zero for all $p < p_c$, and non zero for $p > p_c$, for some value $0 < p_c < 1$ which is called the *critical probability*. The following graph shows the principal features of the graph of $\theta(p)$ against p.

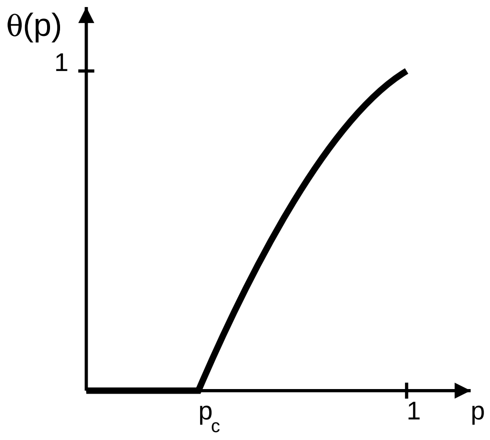


Figure 2. General features of the behaviour of $\theta(p)$ plotted as a function of p. The phase transition occurs at the critical probability p_c. Note that some features of this graph remain conjectures, such as the continuity of $\theta(p)$ at p_c (the possibility of a jump discontinuity has not been ruled out).

1.2. The Phase Transition

An immediately obvious question is what happens to $\theta(p)$ in the vicinity of this critical point p_c? For example is $\theta(p_c)$ zero? This question, trivial though it may seem, remains unanswered for moderate values of the dimension d (it is known to be zero for $d = 2$ and $d \geqslant 19$).

Another conjecture is that in the vicinity of the critical point

$$\theta(p) \approx (p - p_c)^{\beta} \text{ as } p \downarrow p_c.$$

It is not even known how strong is the asymptotic relation implied by the symbol $\approx$, although it is most probably of a logarithmic form, that is

$$\lim_{p \downarrow p_c} \frac{\log \theta(p)}{\log(p - p_c)} = \beta.$$

This constant β is called a *critical exponent* and is believed to be *universal*, that is the same constant applies for all lattices in d dimensional space. Note that this conjecture forces $\theta(p_c)$ to be zero, or else the limit in question would not be defined.

2. Notation

A few details must be cleared up here to avoid confusion later on. In general a graphical notation for open paths on the percolation lattice will be used[1]. The case of bond percolation will be used for the remainder of the essay.

Paths
A *path* connecting x_0 to x_n on the lattice is an alternating sequence $x_0, e_0, x_1, e_1, \ldots, e_{n-1}, x_n$ of distinct vertices x_i and edges e_i between vertices x_i and x_{i+1}. The path is said to be *open* if all the edges e_i are open. The event that a and b are joined by such an open path is denoted

$$a \rightsquigarrow b.$$

Often when describing a more complex event it is important to indicate that two paths in the event are edge disjoint (that is they have no edges in common). This will be expressed by writing each path as

$$\overline{a \quad b}.$$

When a single bond is to be used in a path, it is traversed in a particular direction and so the notation $\overrightarrow{uv}$ will be used for the *directed bond* from u to v (it is broken to indicate that no information is being given as to whether it is open or closed). On the occasions when an undirected bond is referred to, this will be written as (u, v).

Pivotal and Key Bonds
Given a bond configuration, a bond $\overrightarrow{uv}$ is said to be *a pivotal* for the connection $0 \rightsquigarrow x$ if there exists a path from 0 to u, and a path from v to x, but if the bond $\overrightarrow{uv}$ is closed then there is no connection between 0 and x. Note that no information is implied about the actual status of the bond $\overrightarrow{uv}$. This event is expressed diagramatically as:

$$\overline{0 \quad u}\overrightarrow{uv}\overline{v \quad x}.$$

If the pivotal bond is open, then it is possible to go one step further and say that it is a *key* bond for the connection. This is expressed diagramatically as

$$\overline{0 \quad u}\,\boldsymbol{\overline{uv}}\,\overline{v \quad x},$$

The solid bond $\boldsymbol{\overline{uv}}$ indicates that the bond is open. Thus it can be seen how the previous remarks about directionality are significant, since the events

$$\overline{0 \quad v}\overrightarrow{vu}\overline{u \quad x}, \text{ and } \overline{0 \quad u}\overrightarrow{uv}\overline{v \quad x},$$

are distinct. This is advantageous in that it simplifies the notation for summations over, for example, all bonds $\overrightarrow{uv}$ without problems of counting a bond twice.

[1] These notations seem to be non-standard, although based on the notation in (Hara and Slade, 1990a).

Double Connectedness

The point x is said to be *doubly connected* to y if there exist at least two edge disjoint paths from x to y, which will be denoted

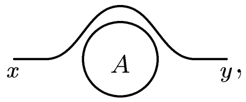
Clusters

The open cluster containing x, denoted $C(x)$ will be defined as the set of all sites connected to x by an open path. The size of the cluster $|C(x)|$ is the number of sites which it contains.

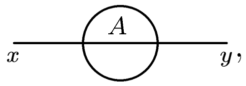
Sometimes restricted clusters of sites connected to x without using a particular bond, or a particular group of sites are of interest. The notation $C_{(u,v)}(x)$ is the set of all sites connected to x without using the (undirected) bond (u, v). Similarly if B is a set of sites, then $C_B(x)$ is the set of sites connected to x without using any site in B.

Avoidance

Later on, it will become useful to consider paths which do, or do not use a set of bonds, especially when describing the lace expansion. The following, fairly obvious graphical notation



denotes a connection from x to y avoiding the sites in the set A i.e. using no bond with an endpoint lying in A, and



denotes a connection between x and y using at least one site in A, and there is no connection from x to y without using a site in A.

3. Basic results

The results presented in this section, are results from the theory of increasing events which will prove useful in the sequel. Lack of space precludes the inclusion of proofs which can be found in (Grimmett, 1989), (Grimmett, 1999).

For percolation, a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ will be used, with sample space $\Omega = \prod_{s \in S}\{0,1\}$, where S is a finite, or countably infinite set, points of which are called *configurations*. For example bond percolation on the hypercubic lattice has $S = \mathbb{E}^d$, and a configuration assigns a value of 1 or 0 to each edge, depending upon whether it is open or closed respectively. The σ-field $\mathcal{F}$ is that subsets of Ω generated by the finite dimensional cylinders. The measure $\mathcal{P}$ is the product measure on $(\Omega, \mathcal{F})$, that is $\mathcal{P} = \prod_{s \in S} \mu_s$, where for each $s \in S$,

$$\begin{aligned} \mu_s(\omega(s) = 0) &= 1 - p(s), \\ \mu_s(\omega(s) = 1) &= p(s). \end{aligned}$$

In this essay the special case where $p(s) = p$ for all $s \in S$ alone will be considered. In this special case, the product measure is denoted $\mathcal{P}_p$.

3.1. Existence of the Phase Transition

The following important theorem implies the existence of a phase transition for dimension greater than or equal to two. Proof may be found in (Grimmett, 1989), (Grimmett, 1999), or (Grimmett, 1996).

Define

$$p_c := \sup\{p : \theta(p) = 0\}.$$

Theorem 3.1.
For dimension $d \geqslant 2$, then $0 < p_c < 1$.

3.2. Increasing Events

To define a partial ordering on Ω, let ω_1 and ω_2 be elements of Ω Then define $\omega_1 \leqslant \omega_2$ if and only if $\omega_1(s) \leqslant \omega_2(s)$ for all $s \in S$.

An event (which is a subset of Ω) is defined to be increasing if $\omega \in A$, and $\omega' \leqslant \omega$ implies that $\omega' \in A$.

An example of an increasing event is $x \leftrightsquigarrow y$ Clearly if this event occurs, making more edges open cannot stop it from occurring, so the event is increasing.

This definition may be extended to a random variable X, which is increasing if $\omega \leqslant \omega'$ implies that $X(\omega) \leqslant X(\omega')$.

3.3. FKG Inequality

Intuitively it seems reasonable to expect some form of positive correlation between increasing events. For example if $x \leftrightsquigarrow y$ then this would be expected to increase the probability of $u \leftrightsquigarrow v$. This form of correlation is expressed by the FKG inequality.

Theorem (FKG Inequality) 3.2.
For increasing random variables X and Y then the following holds.

$$\mathbb{E}_p(XY) \geqslant \mathbb{E}_p(X)\mathbb{E}_p(Y).$$

Corollary 3.3.
For A and B increasing events

$$\mathcal{P}_p(A \cap B) \geqslant \mathcal{P}_p(A)\mathcal{P}_p(B).$$

Proof
Let $X = I(A)$ (here $I(A)$ is the indicator function of A), and $Y = I(B)$ then $\mathbb{E}_p(X) = \mathcal{P}_p(A)$, and $\mathbb{E}_p(Y) = \mathcal{P}_p(B)$ and the result follows from the FKG inequality. □

3.4. BK Inequality

This is a partial converse to the FKG inequality. It concerns itself with disjointly occurring increasing events. First a definition of *disjoint occurrence* is necessary. Given a configuration ω, let $K(\omega)$ be the set of edges which are open in the configuration i.e. $K(\omega) = \{e \in \mathbb{E} : \omega(e) = 1\}$. The event that A and B occur disjointly is denoted $A \circ B$, and is defined to be the event:

$$\begin{aligned} A \circ B = &\{\omega \in \Omega : A \text{ and } B \text{ occur disjointly}\}, \\ = &\{\omega : \exists H \subseteq K(\omega), \text{ such that } \omega' \in A, \omega'' \in B, \\ &\quad \text{where } K(\omega') = H, K(\omega'') = K(\omega) \setminus H\}. \end{aligned}$$

An example of disjoint occurrence in percolation is that of *edge disjoint paths*. Let $A = \{a \leftrightsquigarrow b\}$ and $B = \{c \leftrightsquigarrow d\}$. Then the event $A \circ B$ is the event that there are two edge disjoint paths, one from a to b, and the other from c to d i.e. the event $\{a \longleftrightarrow b$ and $c \longleftrightarrow d\}$.

Theorem (BK Inequality) 3.4.
If A and B are increasing events then

$$\mathcal{P}_p(A \circ B) \leqslant \mathcal{P}_p(A)\mathcal{P}_p(B).$$

3.5. Russo's Formula

Theorem (Russo's Formula) 3.5.
Let E be a finite set, and $\Omega = \{0,1\}^E$, and let A be an increasing event of Ω. Then if $0 < p < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathcal{P}_p(A) = \mathbb{E}_p(N(A)),$$

where the random variable $N(A)$ is the number of edges which are pivotal for the event A.

4. Critical Exponents

As has already been discussed there are singularities in macroscopic functions of the model, such as the percolation probability $\theta(p)$ at the critical probability p_c. It is believed that they exhibit power law behaviour about the critical point, and so critical exponents are introduced to describe this behaviour. There is no general proof of the existence of these exponents, although in the special case of Cayley trees it may be proven, as it will be later for high dimensions.

The same critical exponents are believed to be valid for site or bond percolation, on any lattice in d-dimensional space. This gives rise to the hypothesis of *universality*, which states that critical exponents depend only upon dimension.

The names which are used for the exponents are based on those used in Statistical Mechanics where similar behaviour is observed in, for example, magnetic systems (such as the Ising model, or the q-state Potts model).

4.1. Principal Exponents

Of these critical exponents, there are three principal ones, which are described in this section. As in the introduction, the relation '$\approx$' will be used to denote an imprecise asymptotic relation.

Percolation Probability

The percolation probability has already been mentioned in the introduction:

$$\theta(p) = \mathcal{P}_p(|C(0)| = \infty) \approx (p - p_c)^{\beta} \text{ as } p \downarrow p_c.$$

This corresponds to spontaneous magnetisation in magnetic models.

Mean Cluster Size

The mean cluster size $\chi(p)$ is defined to be the expected number of points in the open cluster containing the origin:

$$\begin{aligned}\chi(p) =& \mathbb{E}_p(|C(0)|) = \sum_x \mathcal{P}_p(0 \leftrightsquigarrow x),\\ =& \text{“}\infty.\mathcal{P}_p(|C| = \infty)\text{”} + \sum_{n=1}^{\infty} n\mathcal{P}_p(|C| = n),\\ \approx& (p_c - p)^{-\gamma} \text{ as } p \uparrow p_c. \end{aligned} \tag{4.1}$$

This corresponds to the magnetic susceptibility in magnetic models.

Cluster Size Distribution

A third principal exponent δ is introduced by

$$\mathcal{P}_{p_c}(|C(0)| = n) \approx n^{-1-1/\delta} \text{ as } n \to \infty.$$

4.2. Other Critical Exponents

In the supercritical phase $\chi(p) = \infty$; but if only the contributions from finite clusters are considered, then the principal critical exponent γ has an associated exponent γ' such that

$$\chi^f(p) = \mathbb{E}_p(|C(0)|; |C(0)| < \infty) \approx (p - p_c)^{-\gamma'} \text{ as } p \downarrow p_c. \tag{4.2}$$

It is conjectured that $\gamma = \gamma'$, in which case as $\chi^f(p) = \chi(p)$ for $p < p_c$ the two relations (4.1) and (4.2) give

$$\chi^f \approx (p - p_c)^{-\gamma} \text{ as } p \to p_c.$$

The remainder of the critical exponents will be introduced as follows. As for $\chi(p)$, in some cases a supercritical analogue (γ') of an exponent from the subcritical phase (γ) may exist. It is however generally conjectured that the two values are equal.

Number of Clusters per Site

Let $\kappa(p)$ be the mean number of clusters per lattice site

$$\kappa(p) = \mathbb{E}(|C(0)|^{-1}; |C(0)| < \infty),$$

this κ is believed to be differentiable twice, but to have a singularity in its third derivative of the form

$$\kappa'''(p) = \begin{cases} |p - p_c|^{-1-\alpha} & \text{as } p \uparrow p_c, \\ |p - p_c|^{-1-\alpha'} & \text{as } p \downarrow p_c. \end{cases}$$

where $-1 \leqslant \alpha, \alpha' < 0$.

Gap Exponents

A sequence of critical exponents can be introduced by considering the ratios of the moments of the cluster size distribution, for $m \geqslant 1$:

$$\frac{\mathbb{E}(|C(0)|^{m+1})}{\mathbb{E}(|C(0)|^m)} \approx (p_c - p)^{-\Delta_{m+1}} \text{ as } p \uparrow p_c.$$

It is conjectured that $\Delta_k = \Delta$ for all $k \geqslant 1$. Analogously if the expectations are restricted to finite clusters

$$\frac{\mathbb{E}(|C(0)|^{m+1}; |C(0)| < \infty)}{\mathbb{E}(|C(0)|^m; |C(0)| < \infty)} \approx |p - p_c|^{-\Delta'_{m+1}} \text{ as } p \downarrow p_c.$$

Similarly it is conjectured that $\Delta'_k = \Delta_k = \Delta$, for all k, and this conjecture is supported by the results of scaling theory.

Cluster Radius and Connectivity Function

At the critical point three exponents are significant δ (already discussed), ρ, and η.

$$\mathcal{P}_{p_c}(\text{rad}(C(0)) = n) \approx n^{-1-1/\rho} \text{ as } n \to \infty,$$

where $\text{rad}(C)$ is defined as $\max\{\|x\|_\infty : x \in C\}$.

$$\mathcal{P}_{p_c}(0 \leftrightsquigarrow x) \approx |x|^{2-d-\eta}.$$

Correlation Length
Let e_1 denote the first unit vector in the standard basis. Define $\tau_n = \mathcal{P}_p(0 \leftrightsquigarrow ne_1)$. By the FKG inequality (and translation invariance).

$$\tau_{m+n} \geqslant \tau_m \tau_n, \tag{4.3}$$

so by the subadditive inequality (proven in Appendix II of (Grimmett, 1989)), the limit $\lim_{m\to\infty} \frac{1}{m} \log \tau_m$ is defined. Hence for $p \leqslant p_c$ it is possible to define a correlation length $\xi(p)$ by

$$\frac{1}{\xi(p)} := -\lim_{m\to\infty} \frac{1}{m} \log \tau_m.$$

The following result can be proven

$$\tau_m \leqslant e^{-\frac{m}{\xi(p)}}.$$

This result gives an example of how the correlation length gives a natural length scale of the system, on which to measure the size of clusters. As p tends to its critical value, this length scale diverges. Hence a critical exponent ν is conjectured

$$\xi(p) \approx (p_c - p)^{-\nu} \text{ as } p \uparrow p_c.$$

In the supercritical phase another definition of the correlation length is needed. There are several possible ways to introduce a 'connectivity' function, one of the most natural being

$$\mathcal{P}_p^f(0 \leftrightsquigarrow x) := \mathcal{P}_p(0 \leftrightsquigarrow x; |C(0)| < \infty).$$

Using this, an analogue of the correlation length can be defined

$$\frac{1}{\xi^f(p)} := -\lim_{m\to\infty} \frac{1}{m} \log \mathcal{P}_p^f(0 \leftrightsquigarrow me_1).$$

The simple multiplicative inequality (4.3) no longer holds, and a more detailed analysis (see (Chayes and Chayes, 1987), or section 4.4.5 of (Hughes, 1996)) is needed to prove that the limit in question exists. It is then postulated that

$$\xi^f(p) \approx (p - p_c)^{\nu'} \text{ as } p \downarrow p_c,$$

where it is conjectured $\nu = \nu'$.

4.3. Scaling Theory

So far a number of critical exponents have been introduced, however it is not believed that they are independent. The following *scaling relations* are widely believed to relate the critical exponents (although this has not been proven). The first group are just statements of the various conjectures about equality of critical exponents, which have already been mentioned when those exponents were introduced.

$$\begin{aligned}\alpha =&\alpha',\\ \gamma =&\gamma',\\ \Delta =&\Delta_k = \Delta'_k \quad \text{for } k \geqslant 1.\end{aligned}$$

The second group are rather more significant relations between the different critical exponents.

$$\begin{aligned}2-\alpha =&\gamma + 2\beta = \beta(\delta+1),\\ \Delta =&\beta\delta,\\ \gamma =&\nu(2-\eta).\end{aligned}$$

Their heuristic derivation comes from scaling theory, which starts by assuming that all the critical exponents conjectured exist. It is then postulated that the behaviour near to the critical point is dominated by a single length scale. This is expressed by introducing the *Stauffer ansatz*[1],

$$\text{singular part of } \mathcal{P}_p(|C(0)|=n) \approx \begin{cases} n^{-\tau} f_+(n^\sigma|p-p_c|) & p>p_c,\\ n^{-\tau} f_-(n^\sigma|p-p_c|) & p<p_c,\end{cases}$$

where the functions f_+ and f_- are smooth functions. In the following, f will be assumed to be chosen appropriately depending upon whether $p > p_c$ or $p < p_c$.

The first relation comes from the list of critical exponents, since there is a direct link between τ in this expression and the critical exponent δ, namely

$$\tau = 1 + \frac{1}{\delta}. \tag{4.4}$$

Given this assumption, the singular part of the kth moment is computed from the ansatz by:

$$\begin{aligned}\text{singular part of } \sum_{n=0}^{\infty} n^k \mathcal{P}_p(|C(0)|=n) \approx& \int_0^\infty n^{k-\tau} f(n^\sigma|p-p_c|)\mathrm{d}n,\\ \approx& \frac{1}{\sigma}|p-p_c|^{(\tau-1-k)/\sigma}\int_0^\infty z^{[(k+1-\tau)/\sigma]-1} f(z)\mathrm{d}z,\end{aligned}$$

where in the last step the change of variable $z = n^\sigma|p-p_c|$ has been made. Noting that many of the critical exponents come from the behaviour near the critical points of such moments, the following identities can be derived, if the Stauffer ansatz is to be consistent with the existence of the critical exponents.

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[1] There is some ambiguity in the definition of the exponents σ, and τ in the literature, some authors preferring to incorporate the correlation length in the ansatz, but this is not significantly different, since the existence of all critical exponents has been assumed, including ν for the correlation length.

Minus first moments
The minus first moment of the cluster size distribution is

$$\sum_n \frac{1}{n}\mathcal{P}_p(|C(0)| = n) = \mathbb{E}_p\left(|C(0)|^{-1}; |C(0)| < \infty\right) = \kappa(p).$$

From the Stauffer ansatz with $k = -1$, it is suggested that:

$$\kappa(p) \approx (p - p_c)^{\tau/\sigma}.$$

The critical exponent α was introduced by suggesting

$$\kappa'''(p) \approx \begin{cases} |p - p_c|^{-1-\alpha} & \text{as } p \uparrow p_c, \\ |p - p_c|^{-1-\alpha'} & \text{as } p \downarrow p_c, \end{cases}$$

so by a fairly large leap of faith

$$2 - \alpha = 2 - \alpha' = \frac{\tau}{\sigma}. \tag{4.5}$$

Zeroth Moments
The zeroth moment is $\sum_n P(|C(0)| = n)$ which is related to the percolation probability by

$$1 - \theta(p) = \sum_n P(|C(0)| = n),$$

hence given that $\theta(p) \approx (p - p_c)^\beta$, it is postulated that

$$\beta = \frac{\tau - 1}{\sigma}. \tag{4.6}$$

First Moments
The first moment $\sum_n n\mathcal{P}_p(|C(0)| = n) = \mathbb{E}(|C(0)|; |C(0)| < \infty)$, directly gives $\chi^f(p)$ (irrespective of whether $p > p_c$, or $p < p_c$), so it is postulated that

$$\gamma = \gamma' = \frac{-\tau + 2}{\sigma}. \tag{4.7}$$

Second and Higher Moments
Taking ratios of moments suggests that

$$\frac{\mathbb{E}(|C(0)|^k; |C(0)| < \infty)}{\mathbb{E}(|C(0)|^{k-1}; |C(0)| < \infty)} \approx |p - p_c|^{-\Delta_k} \text{ as } p \downarrow p_c,$$

which implies

$$\Delta := \Delta_k = \Delta'_k = 1/\sigma. \tag{4.8}$$

Now taking relations (4.4)–(4.8) together the following relations are obtained directly. Multiplying (4.6) and (4.7) by σ and adding yields

$$\sigma = \frac{1}{\beta + \gamma}.$$

Also using (4.4), $\beta\delta = \beta \times 1/(\tau - 1) = 1/\sigma$. Hence from (4.5),

$$\begin{aligned} 2 - \alpha =& \frac{\tau}{\sigma} = \frac{\sigma\beta + 1}{\sigma} \quad \text{from (4.6)}, \\ =& \beta + \frac{1}{\sigma} = \beta + (\gamma + \beta), \\ =& \gamma + 2\beta. \end{aligned}$$

Also the gap exponent satisfies

$$\Delta = \frac{1}{\sigma} = \beta + \gamma = \beta\delta.$$

The final scaling relation may be justified similarly only by the introduction of another *ansatz* such as

$$\mathcal{P}(0 \leftrightsquigarrow x; |C(0)| < \infty) \approx \begin{cases} |x|^{2-d-\eta} g_-(|x|/\xi(p)) & \text{as } p \uparrow p_c \text{ and } x \to \infty, \\ |x|^{2-d-\eta} g_+(|x|/\xi(p)) & \text{as } p \downarrow p_c \text{ and } x \to \infty, \end{cases}$$

where in fact g_- and g_+ are believed to be exponential. This is then used in a similar way to suggest

$$\mathbb{E}_p(|C(0)|; |C(0)| < \infty) \approx \xi(p)^{2-\eta},$$

and therefore the relation $\gamma = \nu(2 - \eta)$ is suggested.

4.4. Hyperscaling Relations

In addition to the scaling relations two *hyperscaling* relations may be suggested, which are believed even less strongly than the scaling relations. These state that for $d \geqslant d_c$, where d_c is some upper critical dimension, which is believed to be six:

$$\begin{aligned} d\rho =& \delta + 1, \\ 2 - \alpha =& d\nu. \end{aligned}$$

Together with the scaling relations these yield

$$2 - \alpha = \gamma + 2\beta = \beta(\delta + 1) = d\nu. \tag{4.9}$$

Thus the picture which this leaves is of a lower critical dimension (which is one for percolation) and for d less than or equal to this lower critical dimension no phase transition occurs in the system. For d greater than this value there is an intermediate region $1 < d < d_c$ where the system undergoes a phase transition, but the values of critical exponents depend upon the dimension d. Finally there is a third region for $d \geqslant d_c$, when the critical exponents are universal and independent of d, and take the same values as the critical exponents for percolation on Cayley Trees. Making this assumption and inserting the values for Cayley Trees (see section 5) in (4.9), one obtains $d(1/2) = 2(1) + 1$ which implies that $d = 6$. This is taken as suggesting that $d_c = 6$, which is backed up by other similarly tenuous arguments from statistical physics. However as the next section shows d_c must be greater than six, and for dimensions greater than nineteen the high dimensional mean field behaviour can be established (see section 7).

The introduction of the hyperscaling relations can be justified (again rather tenuously) by assuming that the correlation length $\xi(p)$ provides the only length scale near the critical point. The quantity κ is analogous to free energy in a magnetic system. The singular part of this may be expected to scale as

$$\kappa \approx (\text{length scale})^d \approx |p - p_c|^{d\nu}.$$

Now it is possible to make a tenuous connection between this and the exponent α which satisfies

$$\kappa \approx |p - p_c|^{-1-\alpha} \text{ as } p \uparrow p_c,$$

suggesting $2 - \alpha = d\nu$, provided $d\nu \leqslant 3$, so that the third derivative does in fact have a singularity.

4.5. Hyperscaling Inequalities

Although the hyperscaling relations have not been proven, various inequalities have been proven (see (Chayes and Chayes, 1987) and (Tasaki, 1987)), which become exact equalities if the hyperscaling hypothesis holds. Examples of these from (Tasaki, 1987) are

$$\begin{aligned}
(d - 2 - \eta) &\geqslant 2\beta, \\
d\nu' &\geqslant \gamma' + 2\beta, \quad d\max(\nu, \nu') \geqslant \gamma + 2\beta \\
d\nu &\geqslant 2\Delta_n - \gamma, \text{ for } n \geqslant 1, \\
d\nu' &\geqslant \Delta'_n + \beta, \quad d\max(\nu, \nu') \geqslant \Delta_n + \beta, \text{ for } n \geqslant 1.
\end{aligned}$$

These imply that the critical exponents cannot simultaneously assume their mean field values in dimension $d < 6$, since the inequalities are inconsistent. This implies that $d_c \geqslant 6$.

4.6. Open Questions

To summarise the results of the previous sections, in general the following results have not been proven rigorously:

- The existence of the critical exponents for $d < 19$,
- The universality hypothesis,
- The scaling relations,
- The hyperscaling relations,

- The conjectured values for the critical exponents[1] for $d = 2$, and the values of these exponents for $6 \leqslant d < 19$ (these are believed to be the mean field values, as for $d > 19$).

[1] It has been conjectured that in $d = 2$ the critical exponents universally assume the following values $\alpha = -2/3$, $\beta = 5/36$, $\gamma = 43/18$, $\delta = 91/5$ and $\nu = 4/3$.

5. Percolation on Trees – Mean Field Values

A Cayley tree is a graph with no closed loops where each vertex (except for one special one called the 'root') has the same co-ordination number (the number of edges which meet there). A binary tree is such a tree where each vertex has co-ordination number 3. Percolation on such trees was first described by Fisher and Essam in 1961 (Fisher and Essan, 1961).

It is possible in this case to prove rigorously the existence of all the critical exponents. However, there is a difficulty in defining some of them such as ν (the exponent for correlation length) since there is no good natural measure of the distance between two sites on the tree. This is usually done by treating the tree as embedded in an infinite dimensional space. Then the distance between sites joined by an n step path may be taken as $\sqrt{n}$. In addition the definitions of the exponents η and ν must be amended (see (Grimmett, 1989) section 8.1). The critical exponents can be shown to take the following values on a tree:

β	1
γ', γ	1
Δ	2
α	-1
δ	2
ρ	$\frac{1}{2}$
η	0
ν	$\frac{1}{2}$

which satisfy the conjectured scaling relations of section 4.3.

As an example, the proof of the existence of β is given in the case of a binary tree.

Theorem 5.1.
For a binary tree $\theta(p) \approx (p - p_c)^{\beta_T}$ *where* $\beta_T = 1$.

Proof
Let C be the open cluster of the tree containing the root. The open portion of the tree may be thought of as a branching process, with a single original parent, each (single) parent may have either zero, one or two offspring. The number in each case is a sample from the binomial distribution $\text{bin}(2, p)$. The cluster C is then finite if and only if the process becomes extinct. Let $G(s)$ be the probability generating function of a typical family size.

$$G(s) = (1-p)^2 s^0 + (1-p)p\frac{1}{2}s^1 + p^2 s^2 = (1-p+ps)^2.$$

The extinction probability is then the smallest non-negative root of $s = G(s)$. This turns out to be 1 if $p \leqslant \frac{1}{2}$ and $(\frac{1-p}{p})^2$ if $p \geqslant \frac{1}{2}$. Hence the percolation probability is given by

$$\theta(p) = \begin{cases} 0 & \text{if } p \leqslant \frac{1}{2}, \\ 1 - (\frac{1-p}{p})^2 & \text{if } p \geqslant \frac{1}{2}. \end{cases}$$

Thus the critical probability for percolation on the Cayley tree is $\frac{1}{2}$. Now differentiate the expression for $\theta(p)$ (for $p \geqslant p_c$) at $p_c = \frac{1}{2}$, to obtain

$$\theta(p) \approx 8(p - p_c)^1 \text{ as } p \downarrow \frac{1}{2}.$$

This proves that $\beta_T = 1$ as required. □

Proofs of the other results may be found in (Grimmett, 1989), or (Hughes, 1996).

5.1. Tree Inequalities

The following inequalities can be proven between the values of the critical exponents in $\mathbb{L}^d$ and those on the tree.

Theorem 5.2.
If any of the exponents β, γ, or δ exists for percolation on $\mathbb{L}^d$, then they must take a value greater than or equal to the corresponding value for percolation on the tree.

For a tree $\gamma_T = 1$, and the proof that $\gamma \geqslant \gamma_T$ is similar to the first half of the proof of Theorem 9.1, which provides the bound

$$\chi(p) \geqslant \frac{1}{2d(p_c - p)} \qquad \text{for } p < p_c,$$

which on comparison with the hypothesis $\chi(p) \approx (p_c - p)^{-\gamma}$ for $p < p_c$ yields the desired inequality $\gamma \geqslant 1$.

6. Low Dimensions

As has been mentioned earlier, rigorous results are known either for $d = 2$ or $d \geqslant 19$. This essay concentrates on the high dimensional case. However a summary of the low dimensional results is presented here. Further details can be found in (Langlands et al., 1992), (Langlands et al., 1994). The following only applies to the two dimensional case, however the lattice will not be fixed as $\mathbb{L}^2$.

6.1. Critical Exponent Results

Kesten has proved the following important results in the two dimensional case. If the critical exponents δ and ρ exist, then all of the other critical exponents (except α) exist and the scaling and hyperscaling relations not involving α hold, namely

$$2\nu = \gamma + 2\beta = \beta(\delta + 1), \quad 2\rho = \delta + 1, \quad \gamma = \nu(2 - \eta).$$

6.2. Conformal Invariance

Let C be a simple closed curve in $\mathbb{R}^2$, and let α and β be arcs of C. Introduce $r > 0$, a *dilation factor* and consider $\mathcal{P}_{p_c}(r\alpha \text{ is connected to } r\beta \text{ in } rC)$. It is believed that the limit as $r \to \infty$ exists and this defines a quantity $\pi(\alpha, \beta; C)$

$$\pi(\alpha, \beta, C) = \lim_{r \to \infty} \mathcal{P}_{p_c}(r\alpha \text{ is connected to } r\beta \text{ in } rC).$$

A graph based model of percolation M is specified by the fundamental data discussed in section 3 (the graph, probability function, configuration space, product measure). An element g of the group $GL(2, \mathbb{R})$ acts on a model M of percolation by sending sites $s \mapsto gs$ and bonds $b \mapsto gb$. The group elements act similarly on events E (such as the crossing event described above). The probability of the event E in model M will be denoted $\pi(E, M)$.

Conjecture (Universality).
For M and M' models of percolation on graphs, then there is an element $g \in GL(2, \mathbb{R})$ such that

$$\pi(E, M') = \pi(E, gM).$$

Note that this is a different assertion from the universality of critical exponents which has been discussed elsewhere.

Conjecture (Conformal Invariance).
For every model M, there is a linear transformation $J = J(M)$ of the plane, establishing a complex structure (multiplication by i is given by $x \mapsto Jx$, and $J^2 = -I$), such that for every function ϕ which is J-holomorphic, or J-antiholomorphic in the interior of C, and continuous up to its boundary,

$$\pi(\phi E, M) = \pi(E, M),$$

for all events E.

An example of establishing a complex structure is given by considering the model M_0 of percolation by site on the square lattice. We may establish a complex structure by identifying one co-ordinate direction with the real axis, and the other with the imaginary axis i.e. by setting

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The ensuing holomorphic functions are the usual ones on the complex plane.

6.3. Cardy's Formula

Cardy (Cardy, 1992) made a remarkable conjecture as to the crossing probability of a region in two dimensional percolation. If the percolation lattice is confined to the upper half plane, then the probability of a crossing between the interval $[x_1, x_2]$ and $[x_3, x_4]$ of the x-axis is given by:

$$\mathcal{P}_{p_c}\left\{[x_1, x_2] \text{ connects to } [x_3, x_4]\right\} = \frac{3\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})^2}\eta^{1/3}{}_2F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \eta\right),$$

where η is the cross-ratio of the four points x_1,x_2, x_3 and x_4, and ${}_2F_1$ is the hypergeometric function, and Γ is the gamma function (for definition see (Abramowitz and Stegun, 1964)).

To apply this to find the crossing probability for a rectangle, consider a Schwartz-Christoffel transformation (see (Nehari, 1952)) of the upper half plane into a rectangle. Let the points corresponding to the vertices be $-1/k$,-1,1 and $1/k$, for a real parameter k, so the points lie on the x-axis, then the transformation is:

$$z \mapsto \int_0^z \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}},$$

Then the cross ratio and aspect ratio of the rectangle are given by

$$\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} = \left(\frac{1-k}{1+k}\right)^2, \qquad r = \frac{2K(k^2)}{K(1-k^2)},$$

where $K(u)$ is the complete elliptic integral.

This value of η may then be substituted into the formula for the crossing probability, giving the probability of crossing a rectangle of aspect ratio r, in the limit as the size of the rectangle becomes infinitely great. This formula may be expressed explicitly in terms of r, see (Ziff, 1995a) and (Ziff, 1995b).

The formula is supported by numerical evidence (such as that of (Langlands et al., 1992) and (Langlands et al., 1994)). I have calculated some additional supporting results which are presented in the appendix, which includes some tabulated values of the crossing probability for a rectangle which was computed from Cardy's formula.

7. High Dimensions

This section presents an overview of results in high dimensional percolation theory and explains the goals of this part of the essay.

The aim is to prove that in high enough dimensions some of the critical exponents of the nearest neighbour percolation model exist. This is a fairly substantial goal and to reach it a number of stages are needed.

(i) Express $\mathcal{P}(0 \leftrightsquigarrow x)$ using convolutions, by using an expansion (the Lace Expansion). In the processes it is necessary to bound the terms in the expansion to be sure that it converges.

(ii) Take the discrete fourier transform of this expression and solve for the fourier transform of $\mathcal{P}(0 \leftrightsquigarrow x)$.

(iii) Use the BK inequality to bound the expression for this fourier transform.

(iv) Using this bound establish that $T(p_c) < \infty$, the so called *triangle condition* holds in sufficiently high dimensions.

(iv) From the finiteness of $T(p_c)$ deduce the main result of high dimension percolation theory, the existence of the critical exponents.

The statement of this main theorem is as follows, although it is conjectured that it is possible to reduce the minimum dimension from nineteen to six (but no lower).

Theorem 7.1.
For all dimensions greater than or equal to nineteen, for the nearest neighbour percolation model, the following hold

$$\begin{aligned}
\chi(p) &\asymp (p_c - p)^{-1} && \text{as } p \uparrow p_c, \\
\theta(p) &\asymp (p - p_c)^{1} && \text{as } p \downarrow p_c, \\
\xi(p) &\asymp (p_c - p)^{-1/2} && \text{as } p \uparrow p_c, \\
\frac{\mathbb{E}(|C(0)|^{m+1}; |C(0)| < \infty)}{\mathbb{E}(|C(0)|^{m}; |C(0)| < \infty)} &\asymp (p_c - p)^{-2} && \text{as } p \uparrow p_c,\ \text{for } m \geqslant 1.
\end{aligned}$$

Here the asymptotics are of the strong form

$$f(x) \asymp g(x) \Rightarrow \exists c_1, c_2 \text{ such that } c_1 g(x) \leqslant f(x) \leqslant c_2 g(x) \text{ for all } x \text{ close to } x_c.$$

In this essay, the first of these results (for $\chi(p)$) will be proven in outline form. Following similar methods, the other results of theorem 7.1 may be proved, apart from that for the correlation length ξ. The proof of this is more involved, but based on proving similar results for e^{mx_1}-weighted quantities, for example the function $e^{mx_1}\mathcal{P}_p(0 \leftrightsquigarrow x)$ replaces $\mathcal{P}_p(0 \leftrightsquigarrow x)$. This proof may be found in (Hara, 1990). The theorem is valid for 'sufficiently high dimension', the best precise form of this condition which has been found so far being $d \geqslant 19$ (proof unpublished).

8. Three Important Lemmas

Three lemmas will be used in the development of the Lace expansion, so they are stated now along with some useful corollaries. Their proof is not fundamental to the structure of the sequel, so it has been deferred until the end (section 12).

Lemma A.
Let A be any non empty set of sites. Let the event E be the event that '0 is connected to u through A, and no pivotal bond for the connection has its first endpoint connected to 0 *through A'. Then*

$$\mathbb{E}_p\left(I(E)I(0 \longrightarrow u \; v \longrightarrow x)\right) = p\mathbb{E}_p\left[I(E)I\left(v \; \boxed{C_{(u,v)}(0)} \; x\right)\right].$$

Corollary A1.

$$\mathcal{P}_p(0 \longrightarrow u \; v \longrightarrow x) = p\mathbb{E}_p\left[I(0 \rightsquigarrow u)\mathcal{P}_p\left(v \; \boxed{C_{(u,v)}(0)} \; x\right)\right].$$

Proof
Take the event E in lemma A to be '0 is connected to u through $\{u\}$'. Then the result follows immediately. $\square$

A similar result holds for $\mathcal{P}_p(0 \longrightarrow u \; v \longrightarrow x)$, without the leading factor of p.

Corollary A2.

$$\mathcal{P}_p(0 \Longleftrightarrow u \; v \longrightarrow x) = p\mathbb{E}_p\left(I(0 \Longleftrightarrow u)\mathcal{P}_p\left(v \; \boxed{C_{(u,v)}(0)} \; x\right)\right).$$

Proof
Take $A = \mathbb{Z}^d$, then the event E requires there to be no pivotal bonds in the connection from 0 to u, which means that there must be two edge disjoint paths between these points, i.e. $0 \Longleftrightarrow u$. $\square$

The following lemma is a technical tool to allow the set of 'closable' bonds to be enlarged arbitrarily in order to complete the proof of the triangle condition in full generality. In our application, we shall take $B = \{z \in \mathbb{Z}^d : \|z\|_\infty \leqslant R\}$.

Lemma B.
Let u be any unit vector. Then for a box $B = \{x \in \mathbb{Z}^d : \|x\|_\infty \leqslant R\} \supset \{0, u\}$,

$$\mathcal{P}_p\left(0 \rightsquigarrow x, \quad u \; \boxed{C_{(0,u)}(x)} \; y\right) \geqslant \alpha(p)\mathcal{P}_p\left(0 \rightsquigarrow x, \quad u \; \boxed{C_B(x)} \; y\right),$$

where $\alpha(p) = \min(p, 1-p)^{\#\text{bonds in } B}$.

Lemma C

The following lemma is used in the conjunction with Lemma B. It should be noted that the event on the left hand side here is the same as that appearing in the right hand side of Lemma B[1].

Lemma C.

$$\mathcal{P}_p\left(0 \rightsquigarrow x,\quad u \; \overset{\frown}{\boxed{C_B(x)}} \; y \right) \geqslant \mathbb{E}_p\left(I(0 \rightsquigarrow x)\,\mathcal{P}_p\left(u \; \overset{\frown}{\boxed{C_B(x)}} \; y \right)\right).$$

[1] Lemma C is used as an equality in (Aizenman and Newman, 1984) (see equation (6.5)), but it would appear that it should in fact be an inequality.

9. The Triangle Condition

As was mentioned in the introduction, as the number of dimensions increases, percolation on the lattice becomes more and more like percolation on a tree. Thus we are led to consider the importance of loops in the lattice, and a measure of this is suggested by the triangle function $T(p)$.

$$T(p) = \sum_{x,y} \mathcal{P}_p(0 \leftrightsquigarrow x)\mathcal{P}_p(x \leftrightsquigarrow y)\mathcal{P}_p(y \leftrightsquigarrow 0),$$

where the sum extends over all vertices x and y. The *triangle condition* is the condition that at the critical point, the triangle function is finite, that is

$$T(p_c) < \infty.$$

The validity of this condition is not immediately obvious, since $\chi(p) = \sum_x \mathcal{P}_p(0 \leftrightsquigarrow x)$ diverges as $p \uparrow p_c$.

Theorem 9.1.
In the nearest neighbour bond percolation model on $\mathbb{Z}^d$, if the triangle condition is satisfied in space of dimension greater than two, then

$$\chi(p) \asymp (p_c - p)^{-1} \text{ as } p \uparrow p_c.$$

The proof of this result uses Russo's formula. Bounds will be obtained for $\chi(p)$, by bounding $\chi'(p)$. The lower bound on χ is fairly straightforward to obtain, but the opposite bound is harder to establish.

Proof
By definition the expected cluster size χ is given by

$$\chi(p) = \sum_x \mathcal{P}_p(0 \leftrightsquigarrow x).$$

It would be useful to apply Russo's formula, but this is not possible directly since the event $0 \leftrightsquigarrow x$ depends upon the status of infinitely many edges. To avoid this problem let, $0 \subset \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \mathbb{Z}^d$, be a sequence of finite subsets of $\mathbb{Z}^d$ such that

$$\bigcup_{n=1}^{\infty} \Lambda_n = \mathbb{Z}^d.$$

Also let $\mathcal{P}_p^{(n)}(x \leftrightsquigarrow y)$ be the probability that x is joined to y using open edges with both endpoints in Λ_n. A restricted version of $\chi(p)$ can also be defined

$$\chi^{(n)}(p) := \max_{x \in \Lambda_n} \sum_{y \in \Lambda_n} \mathcal{P}_p^{(n)}(x \leftrightsquigarrow y).$$

Clearly the following inequality holds

$$\chi(\beta) \geqslant \chi^{(n)}(\beta) \geqslant \sum_{y \in \Lambda_n} \mathcal{P}_p^{(n)}(0 \leftrightsquigarrow y).$$

By application of the monotone convergence theorem

$$\lim_{n \uparrow \infty} \mathcal{P}_p^{(n)}(0 \leftrightsquigarrow y) \uparrow \mathcal{P}_p(0 \leftrightsquigarrow y) \qquad \text{as } n \to \infty.$$

Thus using the bounded convergence theorem the right hand term tends to $\chi(p)$ as $n \to \infty$. Hence $\chi^{(n)}(p) \uparrow \chi(p)$ as $n \to \infty$ (whether or not $\chi(p)$ is finite). But $\chi^{(n)}(p)$ is defined as a maximum over finitely many polynomial functions. These functions will be denoted

$$\chi_w^{(n)}(p) := \sum_{y \in \Lambda_n} \mathcal{P}_p^{(n)}(w \leftrightsquigarrow y), \tag{9.1}$$

for each $w \in \Lambda_n$. Hence $\chi^{(n)}(p)$ is differentiable, except possibly at finitely many points.

As a convention, the value of any restricted function is taken to be zero when any of its arguments is a point outside of Λ_n. It is now possible to apply Russo's formula to (9.1),

$$\frac{\mathrm{d}\chi_w^{(n)}}{\mathrm{d}p} = \sum_x \sum_{u \cdots v} \mathcal{P}_p^{(n)}(w \longleftrightarrow u \cdots v \longleftrightarrow x), \tag{9.2}$$

where the first sum extends over all points u and v which form the endpoints of a bond.

From expression (9.2), an upper bound may be derived for the derivative of $\chi_w^{(n)}(p)$,

$$\begin{aligned} \mathcal{P}_p^{(n)}(w \longleftrightarrow u \cdots v \longleftrightarrow x) \leqslant & \mathcal{P}_p^{(n)}(w \longleftrightarrow u \text{ and edge disjointly } v \longleftrightarrow x) \\ = & \mathcal{P}_p^{(n)}\left((w \leftrightsquigarrow u) \circ (v \leftrightsquigarrow x)\right) \end{aligned}$$

Applying the BK inequality to the right hand side and summing,

$$\begin{aligned} \frac{\mathrm{d}\chi_w^{(n)}}{\mathrm{d}p} \leqslant & \sum_x \sum_{u \cdots v} \mathcal{P}_p^{(n)}(w \leftrightsquigarrow u) \mathcal{P}_p^{(n)}(v \leftrightsquigarrow x), \\ \leqslant & 2d\chi_w^{(n)}(p)\chi^{(n)}(p), \\ \leqslant & 2d\chi^{(n)}(p)^2. \end{aligned}$$

Using this result which is valid for a finite part of the lattice, a bound can be obtained for the derivative of $\chi^{(n)}(p)$ whenever this derivative exists,

$$\begin{aligned} \frac{\mathrm{d}\chi^{(n)}}{\mathrm{d}p} \leqslant & \max_{w \in \Lambda_n} \left\{ \frac{\mathrm{d}}{\mathrm{d}p} \sum_{y \in \Lambda_n} \mathcal{P}_p^{(n)}(w \leftrightsquigarrow y) \right\}, \\ \leqslant & 2d\chi^{(n)}(p)^2. \end{aligned}$$

As $\chi^{(n)}(p)$ is continuous on $[0,1]$, this may be integrated from $p < p_c$ to $p' > p_c$ to give

$$\frac{1}{\chi^{(n)}(p)} - \frac{1}{\chi^{(n)}(p')} \leqslant 2d(p'-p).$$

At this point it is permissible to take the infinite volume limit $n \to \infty$, and then finally taking the limit as $p' \downarrow p_c$, the desired lower bound on χ is obtained, since $\chi^{(n)}(p') \to \chi(p') = \infty$, as $n \to \infty$, so

$$\chi(p) \geqslant \frac{1}{2d(p_c - p)} \qquad \text{for } p < p_c.$$

Note that this was proved entirely without using the triangle condition! For the upper bound on χ more care is needed, but the starting point is the same. To avoid confusion the function χ will be considered in the infinite volume case, although a similar argument to that in the previous part should be used to take the infinite volume limit.

Applying translation invariance[1] to the infinite volume analogue of (9.2) yields

$$\begin{aligned}\frac{\mathrm{d}\chi}{\mathrm{d}p} &= \sum_{x,y} \sum_{|u|=1} \mathcal{P}_p(x \longrightarrow 0 \cdots u \longrightarrow y), \qquad (9.3)\\ &= \sum_{x,y} \sum_{|u|=1} \mathcal{P}_p\left(0 \rightsquigarrow x,\ u \ \boxed{C_{(0,u)}(x)}\ y\right),\end{aligned}$$

where the last assertion follows by rewriting the event in 9.2. Unfortunately by itself this is only sufficient to prove the result for the special case $T(p_c) < 1$ (which in fact never occurs!). A stronger form is needed where instead of considering a single bond being closed as in $C_{(0,u)}$, all the bonds with an endpoint in the box $B(R) = \{x \in \mathbb{Z}^d : \|x\| \leqslant R\}$ are made closed. Let $C_B(x)$ be the set of points reachable from x using no points in B. Using lemma B,

$$\begin{aligned}\frac{\mathrm{d}\chi}{\mathrm{d}p} &\geqslant \alpha(p) \sum_{x,y} \sum_{|u|=1} \mathcal{P}_p\left(0 \rightsquigarrow x,\ u \ \boxed{C_B(x)}\ y\right),\\ &\geqslant \alpha(p) \sum_{x,y} \sum_{|u|=1} \mathbb{E}_p\left[I(0 \rightsquigarrow x)\mathcal{P}_p\left(u \ \boxed{C_B(x)}\ y\right)\right] \qquad \text{by Lemma C.}\end{aligned}$$

The term $\mathcal{P}_p\left(u \ \boxed{C_B(x)}\ y\right)$ is rewritten as

$$\mathcal{P}_p(u \rightsquigarrow y) - \left[\mathcal{P}_p(u \rightsquigarrow y) - \mathcal{P}_p\left(u \ \boxed{C_B(x)}\ y\right)\right].$$

[1] Obviously this does not apply in the finite volume case, but its use is not fundamental to the proof.

So

$$\frac{\mathrm{d}\chi}{\mathrm{d}p} \geqslant \alpha(p) \sum_{x,y} \sum_{|u|=1} \Bigg\{ \mathcal{P}_p(0 \leftrightsquigarrow x)\mathcal{P}_p(u \leftrightsquigarrow y)$$

$$- \mathbb{E}_p \left[I(0 \leftrightsquigarrow x) \left(\mathcal{P}_p(u \leftrightsquigarrow y) - \mathcal{P}_p \left(u \overset{\;C_B(x)\;}{\frown} y \right) \right) \right] \Bigg\}. \tag{9.4}$$

To bound the difference in the second bracket the following consequence of the BK inequality is used.

$$\begin{aligned}
\mathcal{P}_p(u \leftrightsquigarrow y) =& \mathcal{P}_p \left(u \overset{\;A\;}{\frown} y \right) + \mathcal{P}_p \left(u \overset{A}{\longleftrightarrow} y \right), \\
\leqslant& \mathcal{P}_p \left(u \overset{\;A\;}{\frown} y \right) + \sum_{w \in A} \mathcal{P}_p \left(u \text{---} w \text{ and } w \text{---} y \right), \\
\leqslant& \mathcal{P}_p \left(u \overset{\;A\;}{\frown} y \right) + \sum_{w \in A} \mathcal{P}_p(u \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow y) \text{ by BK.}
\end{aligned}$$

Applying this to $A = C_B(x)$, gives

$$\begin{aligned}
\mathcal{P}_p(u \leftrightsquigarrow y) - \mathcal{P}_p \left(u \overset{\;C_B(x)\;}{\frown} y \right) &\leqslant \sum_{w \in C_B(x)} \mathcal{P}_p(u \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow y), \\
&\leqslant \sum_{w \in \mathbb{Z}^d \setminus B} I \left(w \overset{\;B\;}{\frown} x \right) \mathcal{P}_p(u \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow y).
\end{aligned} \tag{9.5}$$

Using this bound (9.5) in the expression (9.4) gives

$$\frac{\mathrm{d}\chi}{\mathrm{d}p} \geqslant \alpha(p) \sum_{x,y} \sum_{|u|=1} \Bigg\{ \mathcal{P}_p(0 \leftrightsquigarrow x)\mathcal{P}_p(u \leftrightsquigarrow y)$$

$$- \sum_{w \in \mathbb{Z}^d \setminus B} \mathbb{E}_p \left[I(0 \leftrightsquigarrow x) I \left(w \overset{\;B\;}{\frown} x \right) \mathcal{P}_p(u \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow y) \right] \Bigg\}. \tag{9.6}$$

If 0 is connected to x, and w is connected to x outside of B, then there must be a point v outside B such that there exist disjoint paths $0 \leftrightsquigarrow v$, $v \leftrightsquigarrow w$, and $v \leftrightsquigarrow x$. Hence using the BK inequality:

$$\mathcal{P}_p \left(0 \leftrightsquigarrow x, \quad w \overset{\;B\;}{\frown} x \right) \leqslant \sum_{v \in \mathbb{Z}^d \setminus B} \mathcal{P}_p(0 \leftrightsquigarrow v)\mathcal{P}_p(v \leftrightsquigarrow w)\mathcal{P}_p(v \leftrightsquigarrow x). \tag{9.7}$$

Putting (9.7) into (9.6) and simplifying gives

$$\frac{\mathrm{d}\chi}{\mathrm{d}p} \geqslant 2d\alpha(p)\chi(p)^2 \left[1 - \max_{|u|=1} \sum_{v,w\in\mathbb{Z}^d\setminus B} \mathcal{P}_p(0 \leftrightsquigarrow v)\mathcal{P}_p(v \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow u)\right].$$

A new form $Q(a,b)$ is defined, which gives the triangle function as a special case $T(p) = Q(0,0)$.

$$Q(a,b) := \sum_{v,w} \mathcal{P}_p(a \leftrightsquigarrow v)\mathcal{P}_p(v \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow b).$$

This Q is a positive definite form, that is

$$\sum_{x,y} f(x)Q(x,y)\overline{f(y)} \geqslant 0,$$

since for any absolutely summable function $f : \mathbb{Z}^d \to \mathbb{C}$,

$$\begin{aligned}\sum_{x,y} f(x)Q(x,y)\overline{f(y)} = \sum_{x,y} g(x)\mathcal{P}_p(x \leftrightsquigarrow y)\overline{g(y)} &= \mathbb{E}_p\left(\sum_{x,y} g(x)I(x \leftrightsquigarrow y)\overline{g(y)}\right), \\ &= \mathbb{E}_p\left(\sum_{\substack{\text{open} \\ \text{cluster } C}} \left|\sum_{x\in C} g(x)\right|^2\right) \geqslant 0,\end{aligned}$$

where

$$g(x) = \sum_a f(a)\mathcal{P}_p(a \leftrightsquigarrow x).$$

Therefore by Schwartz's inequality

$$Q(a,b)^2 \leqslant Q(a,a)Q(b,b) = T(p)^2,$$

so $Q(0,u) \leqslant T(p)$, hence the triangle condition implies finiteness of $Q(0,u)$ for $p \leqslant p_c$. Thus by choosing R so that $B(R)$ is sufficiently large,

$$\sum_{w,v\in\mathbb{Z}^d\setminus B} \mathcal{P}_p(0 \leftrightsquigarrow v)\mathcal{P}_p(v \leftrightsquigarrow w)\mathcal{P}_p(w \leftrightsquigarrow u) \leqslant \frac{1}{2} \text{ for } p \leqslant p_c.$$

Hence the bound

$$\frac{\mathrm{d}\chi}{\mathrm{d}p} \geqslant \frac{1}{2} \times 2d\chi(p)^2\alpha(p) \text{ for } p \leqslant p_c.$$

is obtained, which on integration gives an upper bound on $\chi(p)$. Together with the lower bound obtained earlier

$$\frac{1}{2d(p_c - p)} \leqslant \chi(p) \leqslant \frac{1}{d\int_p^{p_c} \alpha(p)\mathrm{d}p} \text{ for } p \leqslant p_c.$$

for $p \leqslant p_c$. Provided that $p \geqslant \epsilon > 0$, for some ϵ, this may further be simplified using the definition of $\alpha(p)$ (in Lemma B) to

$$\frac{1}{2d(p_c - p)} \leqslant \chi(p) \leqslant \frac{1}{dC(p_c - p)} \text{ for } \epsilon \leqslant p \leqslant p_c,$$

for some constant C. □

10. Lace Expansion and Triangle Condition

In order to link the Lace expansion and the Triangle Condition a key theorem is to be proved. To reach this goal two lemmas are needed.

A new quantity in addition to the triangle function is defined by

$$W(p) := \sum_{x \in \mathbb{Z}^d} |x|^2 \mathcal{P}_p(0 \leftrightsquigarrow x).$$

Lemma 10.1.
Both $T(p)$ and $W(p)$ are continuous functions of p for $p < p_c$.

Proof
For the nearest neighbour model, $\mathcal{P}_p(0 \leftrightsquigarrow x)$ decays exponentially if $\chi(p) < \infty$. It is a standard result that $\mathcal{P}_p(0 \leftrightsquigarrow x)$ is increasing, and continuous in p (for $0 \leqslant p \leqslant 1$). By application of the monotone convergence theorem the continuity of $T(p)$ and $W(p)$ is established. $\square$

Lemma 10.2.
There exist constants k_T and k_W such that for $p \leqslant 1/(2d)$,

$$T(p) \leqslant 1 + \frac{k_T}{d}, \qquad W(p) \leqslant \frac{k_W}{d}.$$

This lemma is proved by comparison with the Gaussian model for the simple random walk. Later on, it is proven in theorem 11.3 that for $p \leqslant 1/(2d)$ the inequality $\mathcal{P}_p(x \leftrightsquigarrow y) \leqslant C_G(x, y)$ holds, where C_G is the Gaussian propagator defined in section 11.2. Hence lemma 10.2 may be proven by bounding the analogous quantities (T_G, W_G) of the Gaussian model, the proof of which may be found in Appendix B of (Hara, 1990).

Lemma[1] 10.3.
Given the constants k_T and k_W of the previous lemma, for $1/(2d) \leqslant p < p_c$, if

$$T(p) \leqslant 1 + \frac{4k_T}{d}, \qquad W(p) \leqslant \frac{4k_W}{d}, \text{ for } p \leqslant \frac{4}{2d}, \tag{10.1}$$

then

$$T(p) \leqslant 1 + \frac{3k_T}{d}, \qquad W(p) \leqslant \frac{3k_W}{d}, \text{ for } p \leqslant \frac{3}{2d}. \tag{10.2}$$

The proof of lemma 10.3 is rather long and technical, a sketch of the key elements of the proof is deferred until after the discussion of the lace expansion, since this forms a key element of the proof.

[1] Two extra conditions are actually required in the full statement of this lemma, they take the form of bounds on W_a and H_{a_1,a_2} as given in the statement of lemma 11.4. Similarly a stronger bound on these quantities may be derived in the conclusion of the lemma. This is omitted here, since the details are technical and only affect the derivation of certain bounds which has been omitted here.

Theorem 10.4.
For suitably high dimensions ($d \geqslant 19$ suffices), the triangle condition is satisfied.

Proof
The conclusion of lemma 10.3 is that there is a region of space in which the pairs $(p, T(p)-1)$ and $(p, W(p))$ may not lie. This is the shaded region in the diagram.

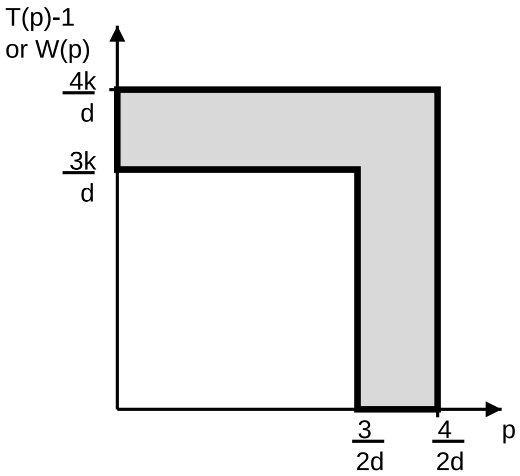


Figure 3. The region of space in which tuples $(p, T(p)-1)$ and $(p, W(p))$ are not allowed to lie as a result of the previous lemma is shown shaded.

But the result of lemma 10.1 implies that $W(p)$ and $T(p)-1$ are continuous for $p < p_c$; they are also zero for $p = 0$; hence to avoid a jump discontinuity it is clear that the condition (10.2) must be satisfied!

This implies

$$T(p) \leqslant 1 + \frac{3k_T}{d} \text{ for all } p < p_c.$$

By application of the monotone convergence theorem coupled with the fact that $\mathcal{P}_p(0 \leftrightsquigarrow x)$ is increasing and continuous the following is obtained

$$T(p_c) = \lim_{p \uparrow p_c} T(p) \leqslant 1 + \frac{3k_T}{d} < \infty.$$

This is the triangle condition. $\square$

11. The Lace Expansion

The lace expansion is used to prove results about percolation in high dimensions d. An insight into what is going on may be gained from the following. As the dimension of the lattice is increased, the percolation on the lattice comes to resemble more and more closely percolation on a tree – that is, if there is a path between two points there is only one such path. If there is a connection between 0 and x then this connection may be split up into a series of doubly connected clusters divided by pivotal bonds.

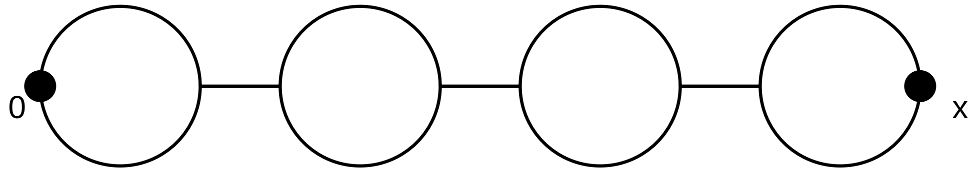


Figure 4. An example of the decomposition of a connection between 0 and x into doubly connected clusters (circles) linked by pivotal bonds (lines).

These doubly connected clusters cannot intersect (or else the bonds between them would not be pivotal). This 'repulsion' of doubly connected clusters is described by the lace expansion[1].

11.1. The Expansion

The first stage in the expansion is to decompose the probability of a connection between vertices 0 and x as follows.

$$\mathcal{P}(0\quad x) = \mathcal{P}(0\quad x) + \sum_{y_1 y_1'} \mathcal{P}(0\quad y_1 y_1'\quad x).$$

Now the corollary A2 allows the second term to be written as

$$\mathcal{P}(0\quad y_1 y_1'\quad x) = p\mathbb{E}_p\left(I(0\quad y_1)\mathcal{P}_p\left(y_1'\ \boxed{C_{(y_1,y_1')}(0)}\ x\right)\right).$$

It is desirable to write this as a convolution, where in this context the convolution of functions f and g is defined by the following

$$f \star g(x) = \sum_y f(x-y)g(y).$$

[1] It is interesting to note that a similar form of lace expansion may be applied to the problem of the self avoiding random walk in high dimensions since the behaviour of a self avoiding random walk becomes more and more like an ordinary random walk as the dimension is increased as in some sense there are more directions to take at each point, so the walk is very unlikely to intersect itself.

To this end the trivial decomposition $a \equiv b - (b-a)$ is used to write

$$\mathcal{P}_p\left(y_1' \xrightarrow{C_{(y_1,y_1')}(0)} x \right) = \mathcal{P}_p(y_1' \rightsquigarrow x) - \left[\mathcal{P}_p(y_1' \rightsquigarrow x) - \mathcal{P}_p\left(y_1' \xrightarrow{C_{(y_1,y_1')}(0)} x \right) \right].$$

For notational simplicity double connectedness probability function g_p is introduced, such that

$$g_p(x) := \mathcal{P}_p(0 \Leftrightarrow x).$$

Let $I(x)$ be the *neighbour* function taking the value 1 at sites x which are nearest neighbours of the origin and zero otherwise.

In order to express the convolution with $\mathcal{P}_p(0 \rightsquigarrow x)$, the notation $\mathcal{P}_p(0 \rightsquigarrow \cdot)$ will be used for the function, such that $\mathcal{P}_p(0 \rightsquigarrow \cdot)(x) = \mathcal{P}_p(0 \rightsquigarrow x)$. Hence

$$\mathcal{P}_p(0 \rightsquigarrow x) = g_p(x) + (g_p \star pI \star \mathcal{P}_p(0 \rightsquigarrow \cdot))(x) - R_p^{(0)}(x), \tag{11.1}$$

where the first remainder term is

$$R_p^{(0)}(x) := p \sum_{y_1 \cdots y_1'} \mathbb{E}_p\left(I(0 \Leftrightarrow y_1) \left[\mathcal{P}_p(y_1' \rightsquigarrow x) - \mathcal{P}_p\left(y_1' \xrightarrow{C_{(y_1,y_1')}(0)} x \right) \right] \right). \tag{11.2}$$

Now to expand this remainder term further a lemma is needed. Frequent reference will be made to the special event $B(x,y;A)$, which is the event[1] that

(i) x is connected to y through A.

(ii) No pivotal bond for the connection from x to y has its first end point connected to x through A.

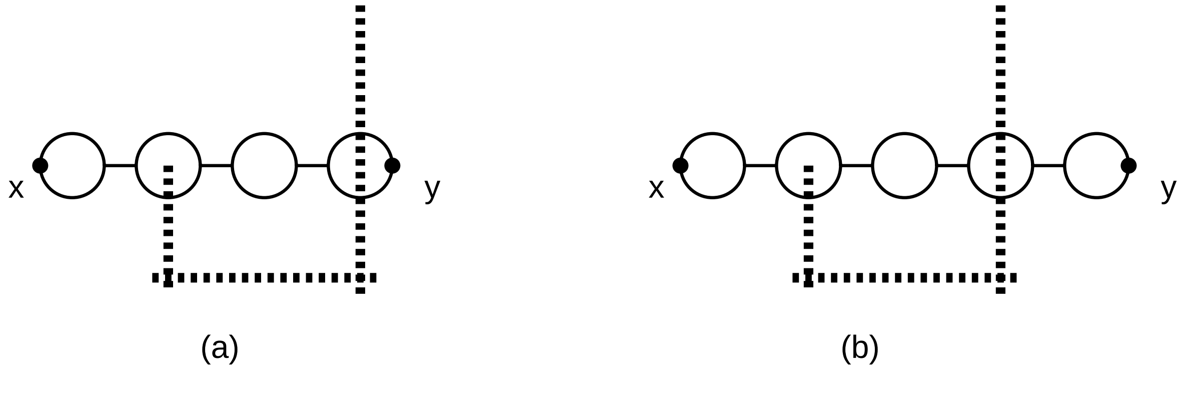


Figure 5. Here single lines denote pivotal bonds for the connection from x to y, and circles clusters of doubly connected sites. The thick dotted line represents the (not necessarily connected) sites of A. In (a) the event $B(x,y;A)$ is taking place, in (b) it is not as there is a pivotal bond 'after' the path has been through A.

[1] This event does not seem to have a standard notation. Hara and Slade in (Hara and Slade, 1994) use E_2.

Lemma 11.1.
For any set A the following holds

$$\mathcal{P}_p(v \sim x) - \mathcal{P}_p\left(v \; A \; x \right) =$$

$$\mathbb{E}_p\left(I(B(v,x;A))\right) + p\sum_{a\,b} \mathbb{E}_p\left[I(B(v,a;A))\mathcal{P}_p\left(b \; \boxed{C_{(a,b)}(v)} \; x\right)\right].$$

Proof
The probability on the left hand side is the probability that v is connected to x through A. If this event occurs then either:

(i) There is no pivotal bond for this connection with its first endpoint connected to v through A, or
(ii) There is such a pivotal bond.

From the above definition (i) is just the occurrence of the event $B(v,x;A)$. In the second case let $a\,b$ be the first pivotal bond for the connection from v to x such that the first endpoint a is connected to v through A. Then the contribution from terms of type (ii) is

$$\sum_{a\,b} \mathcal{P}_p\left(B(v,a;A) \text{ occurs and } v \; a\,b \; x\right)$$

Then by application of lemma A the sum of the two contributions from cases (i) and (ii) gives the right hand side of the lemma. □

Expansion of the first remainder $R_p^{(0)}$
The result of lemma 11.1 is inserted into the expression for the first remainder term in the lace expansion (11.2), superscripts are used on expectations and random quantities for clarity, so for example C^0 is random for $\mathbb{E}^0$, but may be considered as predetermined for $\mathbb{E}^k$, for $k \geqslant 1$.

$$\begin{aligned} R_p^{(0)} =& p\sum_{y_1 y_1'} \mathbb{E}_p\left[I(0 \; y_1)\left(\mathcal{P}_p(y_1' \sim x) - \mathcal{P}_p\left(y_1' \; \boxed{C_{(u,v)}(0)} \; x\right)\right)\right], \\ =& p\sum_{y_1 y_1'} \mathbb{E}_p^0\left[I(0 \; y_1)\mathcal{P}_p\left(B(y_1, x; C^0_{(y_1,y_1')}(0))\right)\right] \qquad (11.3) \\ &+ p^2 \sum_{y_1 y_1'}\sum_{y_2 y_2'} \mathbb{E}_p^0\left[I(0 \; y_1)\mathbb{E}_p^1\left[I(B(y_1', y_2; C^0_{(y_1,y_1')}(0)))\mathcal{P}_p\left(y_2' \; \boxed{C_{(y_2,y_2')}(y_1')} \; x\right)\right]\right]. \end{aligned}$$

Just as in the first stage, the trivial identity $a \equiv b - (b-a)$ is used to split the last term into a convolution term and a next stage remainder term.

$$\mathcal{P}_p\left(y_2' \; \boxed{C_{(y_2,y_2')}(y_1')} \; x\right) = \mathcal{P}_p(y_2' \sim x) - \left[\mathcal{P}_p(y_2' \sim x) - \mathcal{P}_p\left(y_2' \; \boxed{C_{(y_2,y_2')}(y_1')} \; x\right)\right].$$

As the expressions are becoming fairly complicated at this stage it is worthwhile introducing the abbreviations:

$$\Pi_p^{(1)}(x) := p \sum_{y_1 \leftrightarrow y_1'} \mathbb{E}_p^0 \left[I(0 \Leftrightarrow y_1) \mathcal{P}_p \left(B(y_1, x; C^0_{(y_1, y_1')}(0)) \right) \right].$$

$$R_p^{(1)}(x) := p^2 \sum_{y_1 \leftrightarrow y_1'} \sum_{y_1 \leftrightarrow y_2'} \mathbb{E}_p^0 \Bigg[I(0 \Leftrightarrow y_1) \mathbb{E}_p^1 \left[I(B(y_1', y_2; C^0_{(y_1, y_1')}(0))) \right]$$
$$\times \left(\mathcal{P}_p(y_2' \rightsquigarrow x) - \mathcal{P}_p\left(y_2' \; \boxed{C^1_{(y_2, y_2')}(y_1')} \; x \right) \right) \Bigg].$$

Inserting these expressions into (11.3) and then substituting for the remainder term in the first stage convolution equation (11.1) yields

$$\mathcal{P}_p(0 \rightsquigarrow x) = g_p(x) - \Pi_p^{(1)}(x) + ((g_p - \Pi_p^{(1)}) \star pI \star \mathcal{P}(0 \rightsquigarrow \cdot))(x) + R_p^{(1)}(x)$$

This procedure may be continued arbitrarily far, and it is a tedious (although not especially difficult) exercise to prove the full lace expansion theorem stated below. To state it concisely yet more notation is required. Let

$$C^{n-1} := C^{n-1}_{(y_n, y_n')}(y_{n-1}'), \qquad I^n := I(B(y_n', y_{n+1}; C_{n-1})),$$

$$\Pi_p^{(n)}(x) := p^n \sum_{y_1 \leftrightarrow y_1'} \cdots \sum_{y_n \leftrightarrow y_n'} \mathbb{E}_p^0 \left[I(0 \Leftrightarrow y_1) \mathbb{E}_p^1 \left[I^1 \left[\mathbb{E}_p^2 \left[I^2 \cdots \mathbb{E}_p^n [I(B(y_n', x; C^{n-1}))] \right] \cdots \right] \right] \right],$$

$$h_p^{(n)}(x) := g_p(x) + \sum_{j=1}^{n} (-1)^j \Pi_p^{(j)}(x),$$

$$R_p^{(n)}(x) := p^{n+1} \sum_{y_1 \leftrightarrow y_1'} \cdots \sum_{y_{n+1} \leftrightarrow y_{n+1}'} \mathbb{E}_p^0 \Bigg[I(0 \Leftrightarrow y_1) \mathbb{E}_p^1 \Bigg[I^1 \Bigg[\mathbb{E}_p^2 \Bigg[I^2 \cdots \mathbb{E}_p^n \Bigg[I^n$$
$$\times \left(\mathcal{P}_p(y_{n+1}' \rightsquigarrow x) - \mathcal{P}_p\left(y_{n+1}' \; C^n \; x \right) \right) \Bigg]\Bigg]\Bigg]\Bigg]\Bigg]\Bigg].$$

Theorem (Lace Expansion) 11.2.
Given the foregoing definitions, then for $p < p_c$ and $N \geqslant 0$

$$\mathcal{P}_p(0 \rightsquigarrow x) = h_p^{(N)}(x) + \left(h_p^{(N)} \star pI \star \mathcal{P}_p(0 \rightsquigarrow \cdot) \right)(x) + (-1)^{N+1} R_p^{(N)}(x).$$

It is possible to obtain bounds in x-space for each of the terms in this expression by using the BK inequality many times, and hence to obtain bounds for their discrete fourier transforms. The details may be found in section 2.2 of (Hara and Slade, 1990a).

Taking the (discrete) fourier transform of this equation, let $\hat{\tau}(k)$ be the transform of $\mathcal{P}(0 \rightsquigarrow x)$, and solve for $\hat{\tau}(k)$ to get

$$\hat{\tau}(k) = \frac{\hat{g}_p + \sum_{j=1}^{N} (-1)^j \hat{\Pi}_p^{(j)} + (-1)^{N+1} \hat{R}_p^{(N)}}{1 - p\hat{I}\hat{g}_p - p\hat{I} \sum_{j=1}^{N} (-1)^j \hat{\Pi}_p^{(j)}}.$$

11.2. The Simple Random Walk

When constructing bounds in order to prove lemma 10.3 a number of quantities from the simple random walk will be introduced. A simple random walk on the hypercubic lattice can be constructed by building a path which at each vertex takes one of the $2d$ available directions with equal probabilities. A two point function may be defined by

$$C_z(x) := \sum_{\omega:0\to x} z^{|\omega|},$$

where the sum extends over all simple random walks joining the points 0 and x. The quantity $|\omega|$ is the number of steps in the walk ω. As there are $(2d)^n$ simple random walks with n steps on $\mathbb{L}^d$ a trivial bound may be obtained

$$\sum_x |C_z(x)| \leqslant \sum_{n=0}^{\infty}(2d|z|)^n,$$

which implies that the two point function $C_z(x)$ and its fourier transform are finite for $|z| < 1/(2d)$.

An expansion is formed (in some ways analogous to the lace expansion) by conditioning on the first step $(0 \to y)$ in the walk, so

$$C_z(x) = \delta_{0,x} + \sum_{\{y:I(y)=1\}} z \sum_{\omega:y\to x} z^{|\omega|},$$

where $I(x)$ is the nearest neighbour function. This can be expressed as a convolution

$$C_z(x) = \delta_{0,x} + z(I \star C_z)(x).$$

Taking the fourier transform and solving for $\hat{C}_z(k)$ yields

$$\hat{C}_z(k) = \frac{1}{1 - z\hat{I}}.$$

A quantity D is introduced by

$$D(k) := \frac{\hat{I}(k)}{2d} = \frac{1}{d}\sum_{\mu=1}^{d}\cos(k_\mu).$$

From this definition, and the fact that $1 - D(k)$ behaves like k^2 near to 0, a limiting argument shows that $C_{1/(2d)}(x)$ is finite. At this value of $z = 1/(2d)$,

$$C_{1/(2d)}(x) = \sum_{n=0}^{\infty}\mathcal{P}(n \text{ step walk from 0 to } x). \tag{11.4}$$

Thus a *propagator* can be introduced at $z = 1/(2d)$, namely

$$C_G(x,y) := \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \mathrm{d}^d k \, \frac{e^{-ik\cdot(x-y)}}{1 - D(k)}.$$

This is a quantity which can be substituted for τ in many expressions leading to a 'Gaussian' theory, for example

$$T_G := \sum_{x,y} C(0,x)C(x,y)C(y,0),$$

compares with

$$T(p) := \sum_{x,y} \mathcal{P}_p(0 \leftrightsquigarrow x)\mathcal{P}_p(x \leftrightsquigarrow y)\mathcal{P}_p(y \leftrightsquigarrow 0).$$

Similarly to $W(p)$ it is possible to introduce

$$W_G := \sum_x |x|^2 C(0,x)^2.$$

Further useful results bounding quanties in the Gaussian model are given in appendix B of (Hara, 1990). A very simple theorem shows a relationship between the Gaussian and percolation model

Theorem 11.3.
For $0 \leqslant p \leqslant 1/(2d)$, the inequality

$$\mathcal{P}_p(x \leftrightsquigarrow y) \leqslant C_G(x,y)$$

holds.

Proof
Fix p in the interval $[0, 1/(2d)]$. If $x \leftrightsquigarrow y$, then there exists a self-avoiding path of open bonds from x to y. So the probability may be bounded as follows

$$\begin{aligned} \mathcal{P}_p(x \leftrightsquigarrow y) &\leqslant \sum_{\substack{\omega : x \to y \\ \omega \text{ self avoiding}}} \mathcal{P}_p(\omega \text{ uses only open bonds}), \\ &\leqslant \sum_{\omega : x \to y} p^{|\omega|} \leqslant C_G(x,y), \end{aligned}$$

where the final inequality follows from (11.4). $\square$

11.3. Bounds

In order to apply the lace expansion for percolation to the proof of the existence of critical exponents, it is necessary to bound the various terms of the expansion in fourier transform space and hence obtain a bound on $\hat{\tau}(k)$. In this section constraints of space mean that it is not possible to present all the details. Key steps occur in the proof and these have been picked out and described.

From the result of the lace expansion, the fourier transform of $\mathcal{P}(0 \leftrightsquigarrow x)$ is rewritten to look like that of the Gaussian propagator, since it is easier to construct bounds on quantities from the simple random walk model, than upon those of percolation quantities. Noting that

$$\hat{\tau}(k) = \frac{\hat{G}^{(N)}(k)}{1 - 2dpD(k) - \hat{\Xi}^{(N)}},$$

where

$$\hat{G}^{(N)}(k) := \hat{g}_p + \sum_{j=1}^{N} (-1)^j \hat{\Pi}_p^{(j)} + (-1)^{N+1} \hat{R}_p^{(N)},$$

$$\hat{\Xi}^{(N)}(k) := -p\hat{I} + p\hat{I}\hat{g}_p + p\hat{I} \sum_{j=1}^{N} (-1)^j \hat{\Pi}_p^{(j)}.$$

Lemma 11.4.
Let p be fixed, satisfying $1/(2d) \leqslant p < p_c$, and $N \geqslant 0$. Assume that

$$T(p) \leqslant 1 + \frac{4k_T}{d}, \qquad W(p) \leqslant \frac{4k_W}{d}, \text{ for } p \leqslant \frac{4}{2d},$$

where k_W and k_T are the constants of lemma 10.2. and

$$W_a = \sum_x |x|^2 \mathcal{P}_p(0 \leftrightsquigarrow x) \mathcal{P}_p(x \leftrightsquigarrow a) \leqslant \frac{4k_W'}{d} \text{ for } \|a\|_1 \leqslant 2\chi(p)\left((d+2)\ln(5\chi(p)) + 2\ln(d)\right),$$

where k_W' is a universal constant which depends only upon k_W and k_T. Also assume that for

$$\max_{i=1,2} \|a_i\|_1 \leqslant 2\chi(p)\left((5d+2)\ln(5\chi(p)) + 2\ln(d)\right),$$

the following holds

$$H_{a_1,a_2} = \sum_{x,y,z,u,v} |x|^2 \mathcal{P}_p(0 \leftrightsquigarrow x)\mathcal{P}_p(x \leftrightsquigarrow y)\mathcal{P}_p(x \leftrightsquigarrow u)\mathcal{P}_p(0 \leftrightsquigarrow u)$$
$$\times \mathcal{P}_p(0 \leftrightsquigarrow z)\mathcal{P}_p(u \leftrightsquigarrow v)\mathcal{P}_p(v \leftrightsquigarrow y+a_2)\mathcal{P}_p(v \leftrightsquigarrow z+a_1) \leqslant \frac{40k_W}{d}.$$

Then there exists a d_0 independent of p such that for all $d \geqslant d_0$

$$\left|\hat{\Xi}^{(N)}(k)\right| \leqslant \frac{c}{d}, \qquad \left|\partial_\mu^s \hat{\Xi}^{(N)}(k)\right| \leqslant \frac{c'}{d^2}, \quad s = 1, 2. \tag{a}$$

In addition for N sufficiently large (the necessary value of N depends upon both d and p)

$$\hat{F}(k) := 1 - 2dpD(k) - \hat{\Xi}^{(N)}(k) \geqslant \left(1 - \frac{c''}{d}\right)(1 - D(k)), \tag{b}$$

$$|\hat{G}^{(N)}(k) - 1| \leqslant \frac{c}{d}, \qquad |\partial_\mu^s \hat{G}^{(N)}(k)| \leqslant \frac{c'}{d^2}, \quad s = 1, 2, \tag{c}$$

and

$$0 \leqslant \hat{\tau}(k) \leqslant \frac{1}{1 - D(k)}\left(1 + \frac{c'''}{d}\right), \tag{d}$$

where the constants c,c',c'', and c''' depend only on k_T and k_W.

The proof of lemma 11.4 follows from a series of bounds obtained from diagram expansions and may be found in (Hara and Slade, 1990a). Using this lemma the following sketch shows the structure of the proof of lemma 10.3. Note that the last two conditions (bounds on W_a and H_{a_1,a_2}) in lemma 11.4 are highly technical and for this sketch proof are completely ignored.

Proof of Lemma 10.3 (sketch)

Choose N sufficiently large that the conditions of lemma 11.4 are satisfied. The proof is split into a number of parts, each proving one of the inequalities in the conclusion of lemma 10.3.

(a) $p \leqslant \frac{3}{2d}$. By (c) of the lemma for large enough d

$$1 - 2dp - \hat{\Xi}^{(N)}(0) = \hat{G}^{(N)}(0)/\hat{\tau}(0) = \hat{G}^{(N)}(0)/\chi(p) \geqslant 0,$$

so by part (a),

$$2dp \leqslant 1 - \hat{\Xi}^{(N)}(0) \leqslant 1 + \frac{c}{d},$$

whence, for sufficiently large d ($d \geqslant c/2$ suffices), $p \leqslant \frac{3}{2d}$.

(b) $T(p) \leqslant 1 + \frac{3k_T}{d}$. By fourier transform results

$$\begin{aligned} T(p) =& \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}^d k\, \hat{\tau}(k)^3, \\ =& 1 + \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}^d k\, \left[(2 + \hat{\tau}(k))(\hat{\tau}(k) - 1)^2\right]. \end{aligned} \tag{11.5}$$

The first bracket $(2 + \hat{\tau}(k))$ may be bounded using part (d) of the lemma, for $d \geqslant 4c'''$

$$0 \leqslant \hat{\tau}(k) + 2 \leqslant 2 + \frac{1}{1 - D(k)}\left(1 + \frac{c'''}{d}\right) \leqslant 2 + \frac{5}{4}\frac{1}{1 - D(k)}. \tag{11.6}$$

From the *lace expansion* bound on $\hat{\tau}(k)$,

$$\hat{\tau}(k) - 1 = \frac{2dpD(k) + \hat{G}^{(N)}(k) - 1 + \hat{\Xi}^{(N)}(k)}{1 - 2dpD(k) - \hat{\Xi}^{(N)}(k)} = \frac{2dpD(k) + \hat{G}^{(N)}(k) - 1 + \hat{\Xi}^{(N)}(k)}{\hat{F}(k)}.$$

Now using (a) – (c) of the lemma the following bound can be established for the second bracket in (11.5), for sufficiently large d (using the Schwartz inequality).

$$(\hat{\tau}(k)-1)^2 \leqslant \frac{2[(2dpD(k))^2+(|\hat{G}(k)-1|+|\hat{\Xi}^{(N)}(k)|)^2]}{\hat{F}(k)^2},$$
$$\leqslant 20\frac{D(k)^2+c'/d^2}{(1-D(k))^2}.$$

Now putting these two bounds together into (11.5) (using the fact that $(1-D(k))^{-m}$ has an integral which is bounded uniformly in $d \geqslant 7$ for $m = 1, 2, 3$);

$$T \leqslant 1+20\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}^d k \left(2+\frac{5}{4}\frac{1}{(1-D(k))}\right)\frac{D(k)^2}{(1-D(k))^2}+\frac{c''}{d^2},$$
$$\leqslant 1+25T_G+\frac{c''}{d} \leqslant 1+\frac{k_T}{d}+\frac{c''}{d^2} \leqslant 1+\frac{3k_T}{d},$$

which holds for sufficiently large d, using the fact that $C_G(0,0)-1 \leqslant c/d$.

(c) $W(p) \leqslant \frac{3k_W}{d}$. Using Parseval's theorem, $W(p)$ can be written as

$$W(p) := \sum_x |x|^2 \mathcal{P}_p(0 \rightsquigarrow x)^2 = \sum_{\mu=1}^{d} \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}^d k\ (\partial_\mu \hat{\tau}(k))^2 .$$

Now differentiating the result for $\hat{\tau}(k)$ from the lace expansion and substituting in the expression for $W(p)$ one obtains

$$W(p) \leqslant \sum_{\mu=1}^{d} \frac{3}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}^d k \left(\frac{9[\hat{G}^{(N)}(k)\partial_\mu D(k)]^2}{\hat{F}(k)^4}+\frac{[\hat{G}^{(N)}(k)\partial_\mu \hat{\Xi}^{(N)}(k)]^2}{\hat{F}(k)^4} + \frac{[\partial_\mu \hat{G}^{(N)}(k)]^2}{\hat{F}(k)^2}\right). \tag{11.7}$$

Now bound the powers of $\hat{F}(k)$ occurring in the denominators using part (b) of the lemma. By part (c) the first term of the integrand (including the summation) may be bounded by $30W_G$ for sufficiently large d, and the third term is bounded by c/d^3. To bound the second term note that by symmetry $\partial_\mu \hat{\Xi}^{(N)}(k)$ must equal zero for any k with zeroth μth component. Let $\tilde{k}$ be k with the μth component set to zero. By Taylor's theorem there is a point k^* on the line segment joining k and $\tilde{k}$ such that

$$\partial_\mu \hat{\Xi}^{(N)}(k) = \partial_\mu \hat{\Xi}^{(N)}(k) - \partial_\mu \hat{\Xi}^{(N)}(\tilde{k}) = k_\mu \partial_\mu^2 \hat{\Xi}^{(N)}(k^*).$$

Now using (a) and (c) the second term of (11.7) may be bounded by

$$\frac{[\hat{G}^{(N)}(k)\partial_\mu \hat{\Xi}^{(N)}(k)]^2}{\hat{F}(k)^4} \leqslant \frac{c}{d^4}\frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \mathrm{d}^d k\, \frac{k^2}{[1-D(k)]^4}.$$

Noting that

$$\frac{\pi^2}{2}[1-D(k)] \geqslant \frac{k^2}{d},$$

the second term of (11.7) can hence be bounded by c'/d^3. For sufficiently large d, inserting these bounds in (11.7) gives the desired bound on $W(p)$. □

12. Proofs of Three Important Lemmas

These lemmas were introduced in section 8, without proof. The proofs are presented here.

Lemma A.

Let A be any non empty set of sites. Let the event E be the event that '0 is connected to u through A, and no pivotal bond for the connection has its first endpoint connected to 0 through A' (this is the same as the event $B(0,u;A)$ described in section 11.1). Then

$$\mathbb{E}_p\left(I(E)I(0 \quad u\ v \quad x)\right) = p\mathbb{E}_p\left[I(E)I\left(v \quad C_{(u,v)}(0) \quad x\right)\right].$$

Proof

First note that both the event E and the event that '(u,v) is a pivotal bond for the connection 0 to x' are independent of the status of the bond (u,v) (whether it is open or not), hence

$$\mathbb{E}_p\left(I(E)I(0 \quad u\ v \quad x)\right) = p\mathbb{E}_p\left[I(E)I(0 \quad u\ v \quad x)\right].$$

Now conditioning on the cluster $C_{(u,v)}(0)$, we obtain

$$\begin{aligned}
&\mathbb{E}_p\left(I(E)I(0 \quad u\ v \quad x)\right) \\
&\qquad = p \sum_{\substack{\{S:0\in S\} \\ S \text{ finite}}} \mathbb{E}_p\left[I(E \text{ occurs}, 0 \quad u\ v \quad x \text{ and } C_{(u,v)}(0) = S)\right]. \qquad (12.1)
\end{aligned}$$

In (12.1) it is possible to replace the statement $0 \quad u\ v \quad x$ by 'v connects to x outside of S' (since suppose a bond with an endpoint in S were used in the connection from v to x, then there is a connection from 0 to x irrespective of the bond $u\ v$ which contradicts the pivotal nature of the bond), and so the right hand side of (12.1) becomes

$$p \sum_{\substack{\{S:0\in S\} \\ S \text{ finite}}} \mathbb{E}_p\left[I\left((E \text{ occurs}, v \quad S \quad x, C_{(u,v)}(0) = S\right)\right].$$

The event E requires that no pivotal bonds for the connection from 0 to x have their first endpoint connected to 0 through A, so it is determined by bonds with at least one endpoint in $C_{(u,v)}(0)$. Also the event $C_{(u,v)}(0) = S$ depends only upon the status of bonds in S. Hence the event $\{E \text{ and } C_{(u,v)}(0) = S\}$ is independent of the event v connects to x avoiding S. Therefore the right hand side of (12.1) becomes

$$p \sum_{\substack{\{S:0\in S\} \\ S \text{ finite}}} \mathbb{E}_p\left[I\left(E \text{ occurs and } C_{(u,v)}(0) = S\right)\right]\mathcal{P}_p\left(v \quad S \quad x\right).$$

But the probability term can be taken inside the expectation and performing the sum over S we get:

$$\mathbb{E}_p\left(I(E)I(0 \quad u\ v \quad x)\right) = p\mathbb{E}_p\left[I(E)\mathcal{P}_p\left(v \quad \boxed{C_{(u,v)}(0)} \quad x\right)\right].$$

□

Lemma B.
Let u be any unit vector. Then for a box $B(R) = \{x \in \mathbb{Z}^d : \|x\|_\infty \leqslant R\} \supset \{0, u\}$,

$$\mathcal{P}_p\left(0 \quad x, \quad u \quad \boxed{C_{(0,u)}(x)} \quad y\right) \geqslant \alpha(p)\mathcal{P}_p\left(0 \quad x, \quad u \quad \boxed{C_B(x)} \quad y\right),$$

where $\alpha(p) = \min(p, 1-p)^{\#\text{bonds in } B}$.

Proof
Define three events E, F, G as follows

$$E = \left\{0 \quad x, \quad u \quad \boxed{C_{(0,u)}(x)} \quad y\right\},$$

$$F = \left\{0 \quad x, \quad u \quad \boxed{C_B(x)} \quad y\right\},$$

$$G = \{C(x) \cap B \neq \emptyset, C(y) \cap B \neq \emptyset, \text{ and } C_B(x) \cap C_B(y) = \emptyset\}.$$

It is straightforward to see that $E \subseteq F \subseteq G$, and hence

$$\mathcal{P}_p(G) \geqslant \mathcal{P}_p(F), \qquad \mathcal{P}_p(E) = \mathcal{P}_p(G)\mathcal{P}_p(E|G). \tag{12.2}$$

The event G depends only upon bonds with at least one endpoint not in B. Hence, given that for the configuration ω, the event G occurs (i.e. $\omega \in G$), provided $d > 1$ it is possible to construct at least one configuration of bonds with both endpoints in B, such that replacing the configuration of these particular bonds which occurs in ω, with this new configuration, means that the event E occurs. The diagram shows an example of this process.

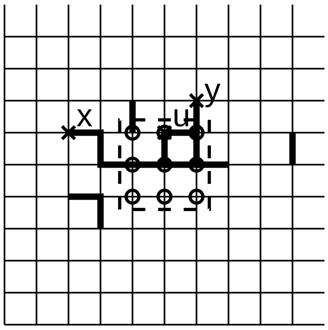


Figure 6. The box $B(1)$ is shown bounded by a dashed line, and all the sites within it are marked by circles. Thicker lines indicate the presence of an open bond. G is shown occurring, i.e. x connects to a site in B, and y connects to a site in B, but x and y are not connected outside of B.

As B is finite, so is the number of edges with both endpoints in B, and hence

$$\mathcal{P}_p(\text{a configuration inside } B) \geqslant \min(p, 1-p)^{\#\text{ of edges in } B} = \alpha(p),$$

So $\mathcal{P}_p(E|G) \geqslant \alpha(p)$. Inserting this in equation (12.2) implies that

$$\mathcal{P}_p(E) = \mathcal{P}_p(E|G)\mathcal{P}_p(G) \geqslant \mathcal{P}_p(E|G)\mathcal{P}_p(F) \geqslant \alpha(p)\mathcal{P}(F).$$

But from the definitions of E and F, this is just the statement of the lemma. $\square$

In the case of $B = B(R)$, then the number of points in B is $(2R+1)^d$ and a bound on the number of edges in E_B is $d(2R+1)^d$, so $\alpha(p) = \min(p, 1-p)^{d(2R+1)^d}$.

Lemma C.

$$\mathcal{P}_p\left(0 \rightsquigarrow x,\quad u \; \boxed{C_B(x)} \; y\right) \geqslant \mathbb{E}_p\left(I(0 \rightsquigarrow x)\mathcal{P}_p\left(u \; \boxed{C_B(x)} \; y\right)\right).$$

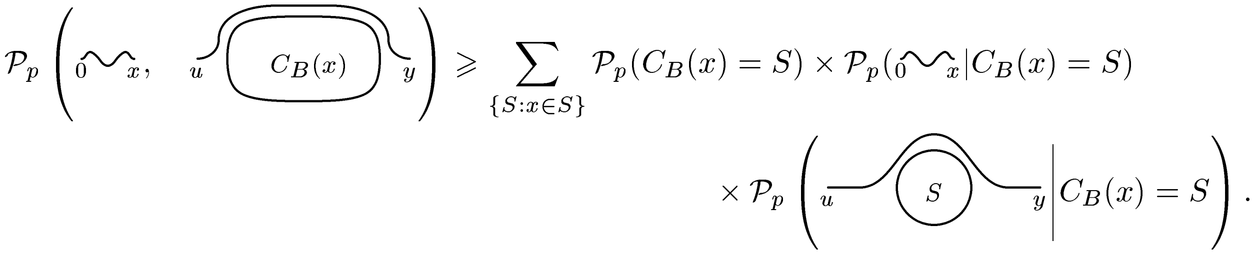
Proof

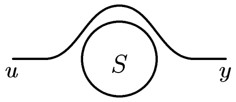
Conditioning on the random set $C_B(x)$.

$$\mathcal{P}_p\left(0 \rightsquigarrow x,\quad u \; \boxed{C_B(x)} \; y\right) = \sum_{\{S: x \in S\}} \mathcal{P}_p(C_B(x) = S)$$

$$\times \mathcal{P}_p\left(0 \rightsquigarrow x,\quad u \; S \; y \,\middle|\, C_B(x) = S\right).$$

The two events in the second probability term are not independent, but they depend only upon bonds not touching S and bonds connecting S to B. Restricted to this set of bonds the two events are increasing and so the FKG inequality may be applied.



Finally independence of the events $C_B(x) = S$ and  allows the desired conclusion to be attained. □

13. Appendix

This appendix presents some numerical evidence which I have computed in favour of the hypothesis of conformal invariance and more specifically Cardy's formula, which was described in Section 6.3.

13.1. Crossing Probability for a Rectangle

The first set of numerical experiments aim to determine the crossing probably from one edge to an opposite one of a rectangle in $\mathbb{L}^2$ for a site percolation mode. Site percolation was chosen since this simplified the computations. One million configurations were generated for each rectangle size, and in each case a value of p_c of 0.5927439 was used[1] (to save computer time, these configurations were in fact generated as required, so it was never decided if unreached bonds were open or not). For each configuration an attempt was made to cross from the left hand edge to the right hand edge via a 'wetted sites' algorithm[2] (which has an empirically determined complexity proportional to the number of sites in the rectangle raised to the 0.946th power). The number of configurations for which this crossing was successful was then recorded, and this used to compute an estimate of the crossing probability. For a million configurations, assuming that the random number generator is perfect, this should give a 95% confidence interval of ± 0.00098 in the worst case ($p = 0.5$).

13.2. Comments on Results

Note that when compared with the data in (Langlands et al., 1994) there is one particularly striking difference, namely in their results table (3.2) all of the simulated values for the horizontal crossing ($\hat{\pi}_h$) lie above the predictions from Cardy's formula (π_h^{cft}). I suspect that this is due to deficiencies in the random number generator which was used in those simulations (a linear congruential generator). For the above results `dprand()` written by Nick Maclaren was used (available from `cus.cam.ac.uk` by anonymous ftp). The above computations took just under three days on a network of 34 Pentium II processors[3].

[1] Various estimates for this value have been given, see (Hughes, 1996) page 184, for some examples. A better estimate would seem to be that of (Ziff, 1994), which gives $p_c = 0.5927460 \pm 0.0000005$. The particular value used in these simulations was taken from (Langlands et al., 1994).

[2] An alternative algorithm, based on attempting to construct the boundary of a percolation cluster via a type of self avoiding walk, is given in (Ziff et al., 1984), which could possibly be more efficient.

[3] Peter Benie made many helpful suggestions for improving the speed of the program when running on modern hardware, not suffering from a shortage of RAM.

Crossing Probability for a Rectangle				
Width	Height	Aspect Ratio	Simulated	Cardy's Formula
1000	1000	1.0000	0.499791	0.500000
1025	975	1.05128	0.473003	0.473997
1050	950	1.10526	0.447092	0.448046
1080	930	1.16129	0.422270	0.422583
1105	905	1.22099	0.397077	0.397021
1135	880	1.28977	0.369689	0.369465
1160	860	1.34884	0.347837	0.347322
1190	840	1.41667	0.323950	0.323520
1220	820	1.4878	0.299734	0.300306
1250	800	1.5625	0.276987	0.277715
1285	780	1.64744	0.254939	0.254083
1315	760	1.73026	0.233633	0.232977
1350	740	1.82432	0.211164	0.211124
1385	725	1.91034	0.192903	0.192937
1420	705	2.01418	0.173399	0.173058
1455	685	2.12409	0.154467	0.154243
1490	670	2.22388	0.139058	0.138938
1530	655	2.33588	0.123814	0.123562
1570	640	2.45312	0.109851	0.109286
1610	620	2.59677	0.094281	0.094023
1650	605	2.72727	0.081963	0.082013
1690	590	2.86441	0.071104	0.071042
1735	575	3.01739	0.060842	0.060526
1775	565	3.14159	0.053138	0.053144
1820	550	3.30909	0.044863	0.044594
1870	535	3.49533	0.036933	0.036692
1915	520	3.68269	0.030278	0.030156
1965	510	3.85294	0.025503	0.025231
2015	495	4.07071	0.020337	0.020086
2065	485	4.25773	0.016841	0.016514
2115	470	4.50000	0.012887	0.012813
2170	460	4.71739	0.010452	0.010205
2225	450	4.94444	0.008106	0.008045
2280	440	5.18182	0.006356	0.006274
2340	425	5.50588	0.004491	0.004469
2400	415	5.78313	0.003390	0.003343
2460	405	6.07407	0.002538	0.002465
2520	395	6.37975	0.001832	0.001790
2585	385	6.71429	0.001305	0.001261
2650	375	7.06667	0.000831	0.000872
2720	370	7.35135	0.000668	0.000647

13.3. Graphical Comparison

To illustrate the closeness of the agreement between simulation and computation through Cardy's formula, the following graph shows a curve of results obtained from Cardy's formula (numerical integration was used to compute the special functions). The simulation results have been superimposed as crosses (the error bars are so small that they would be invisible on such a graph, so they have been omitted).

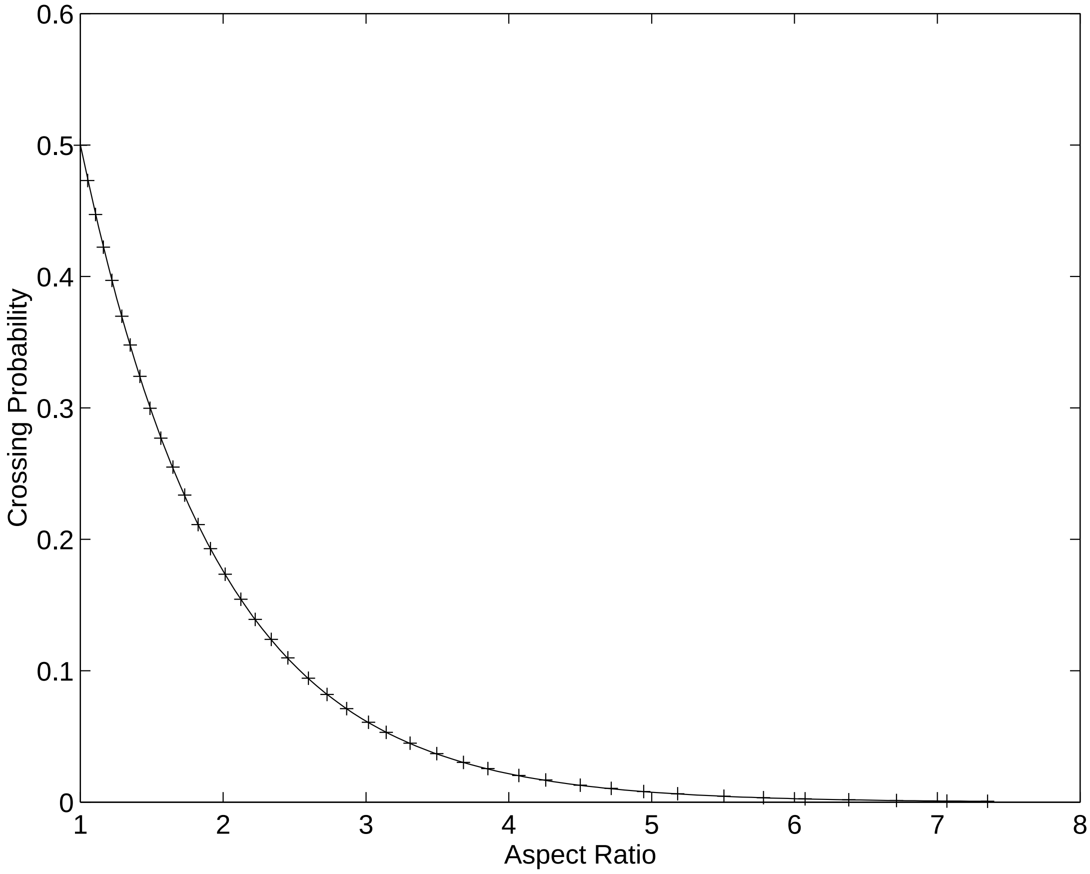


Figure 7. Comparison of simulated results and Cardy's prediction of crossing probabilities for site percolation on rectangles. Aspect ratio is plotted along the horizontal axis, and the prediction forms the solid line. Simulated results are shown as crosses.

13.4. The Triangle Conjecture

Another demonstration of conformal invariance is provided by a rather nice problem, the triangle crossing conjecture.

Conjecture 13.1.

Consider a triangular section ABC of the triangular lattice, with sides of length n. Now take the sections A_x, of length xn starting at vertex A along side AB. Then the following *holds:*

$$\lim_{n\to\infty} \mathcal{P}_p(A_x \text{ is connected to } BC \text{ in triangle}) \to x.$$

This can be verified from Cardy's formula and conformal invariance. Consider the Schwartz-Christoffel transform between the upper half plane and the equilateral triangle ABC. Let the points -1, 0 and $+1$ on the real axis be mapped to the vertices B,C, and A respectively. The transformation is then given by:

$$f(z) = \int_0^z \frac{1}{w^{2/3}(w-1)^{2/3}(w+1)^{2/3}}$$

The image of the upper half plane under $f(z)$ is the triangle with vertices 0,

$$\int_0^1 \frac{1}{w^{2/3}(w-1)^{2/3}(w+1)^{2/3}} \approx -2.103 - 3.643i,$$

and

$$\int_0^{-1} \frac{1}{w^{2/3}(w-1)^{2/3}(w+1)^{2/3}} \approx 2.103 - 3.643i.$$

Now take another point y on the real axis satisfying $y \geqslant 1$ or $y \leqslant -1$ and let its image be the other end of the portion A_x. First the length of this is computed as a fraction of the length of the side, giving x of the conjecture. Then the anharmonic ratio of all four points on the real axis is computed

$$\eta = \frac{(-1-0)(1-y)}{(-1-1)(0-y)},$$

which may be inserted into Cardy's formula, and evaluated numerically. Note that this would be much simpler if an analytic inversion of the Schwartz-Christoffel transform could be used. NAg library routines are used to evaluate the special functions (hypergeometric, gamma, and Schwartz-Christoffel integral) providing results which agree to six decimal places (the accuracy used for the numerical computations). It would therefore appear that proving this exactly should be an exercise in manipulation of special functions. Thus Cardy's formula and conformal invariance would seem to confirm the conjecture. The table shows some of the computed results for various values of x. It is an extract from a much larger table.

Crossing Probabilitity for Triangle	
x	Cardy Formula Crossing Probability
0.02085279513730	0.02085281535983
0.09702920061253	0.09702920913696
0.12194495728577	0.12194494903088
0.17508969574916	0.17508968710899
0.20684047765034	0.20684047043324
0.23057401652073	0.23057401180267
0.24983962963429	0.24983958899975
0.26617980842160	0.26617980003357
0.28042275997541	0.28042271733284
0.29307130306207	0.29307126998901
0.30445674885768	0.30445671081543
0.31481048524632	0.31481042504311
0.32430135595284	0.32430133223534
0.33305697146615	0.33305695652962
0.34117648992543	0.34117650985718
0.34873872207235	0.34873870015144
0.35580751565984	0.35580748319626
0.36243543952136	0.36243546009064
0.36866647577520	0.36866641044617
0.37453774850322	0.37453770637512
0.38008108239540	0.38008108735085
0.38532388125871	0.38532385230064
0.39029006320389	0.39029005169868
0.39500062742532	0.39500060677528
0.39947407928593	0.39947405457497
0.40372691208009	0.40372687578201
0.40777391876253	0.40777391195297

13.5. Numerical Simulation

The simulation used in the previous part was easily extended to cover site percolation on a triangular lattice[1]. The number of configurations used for each value of x varies, but a 95% confidence interval was computed for the crossing probability, based upon the results of the simulation. The results produced[2] strongly confirm the hypothesis that as the side

[1] This was facilitated by using the lattice embedding suggested in figure 2.5 of (Kesten, 1982).

[2] The original results were computed on a small network of workstations (two Sun IPX workstations, HP9000/835, HP9000/400t). These were then extended by further computations on a larger network of 34 Pentium II processors.

length tends to infinity the crossing probability is x. The computations were performed on a triangles of side 1000, and on those of side 5000. Typically 10000 configurations took just over two days to compute on a Sun IPX.

Results for Side Length 1000

Crossing Probabilitity for Triangle (side 1000)			
x	Simulated Crossing Probability	95% Confidence Interval	Number of Configurations
0.10	0.100750	(0.099430821, 0.102069179)	200000
0.20	0.199665	(0.197913025, 0.201416975)	200000
0.25	0.248890	(0.246210143, 0.251569857)	100000
0.30	0.299485	(0.297477584, 0.301492416)	200000
0.40	0.399802	(0.398444184, 0.401159816)	500000
0.50	0.499312	(0.497926072, 0.500697928)	500000
0.60	0.599759	(0.598685106, 0.600832394)	800000
0.70	0.698235	(0.697073509, 0.699396491)	600000
0.75	0.749760	(0.747075301, 0.752444699)	100000
0.80	0.799501	(0.798673931, 0.800328292)	900000
0.90	0.899015	(0.898354728, 0.899675272)	800000

Results for Side Length 5000

Crossing Probabilitity for Triangle (side 5000)			
x	Simulated Crossing Probability	95% Confidence Interval	Number of Configurations
0.10	0.099980	(0.098120746, 0.101839254)	100000
0.20	0.200420	(0.198665544, 0.202174456)	200000
0.30	0.297100	(0.294267603, 0.299932397)	100000
0.40	0.400500	(0.399539601, 0.401460399)	1000000
0.50	0.500711	(0.499731001, 0.501690999)	1000000
0.60	0.600780	(0.597744572, 0.603815428)	100000
0.70	0.700850	(0.698843228, 0.702856772)	200000
0.80	0.799830	(0.799003328, 0.800656672)	900000
0.90	0.899976	(0.899387937, 0.900564063)	1000000

14. References

The following references were all consulted at some stage in writing this essay, although in many cases only for small pieces of information. The principal references were (Grimmett, 1989),(Hughes, 1996), (Grimmett, 1996), (Hara and Slade, 1990a), and (Hara and Slade, 1994).

Abramowitz, M. and Stegun, A. (1964). *Handbook of Mathematical Functions.* Dover.

Aharony, A. and Hovi, J.-P. (1994). Comment on "Spanning probability in 2D percolation". *Phys. Rev. Lett.*, 72(12):1941.

Aizenman, M. and Fernández, R. (1986). On the critical behaviour of the magnetisation in high dimensional Ising models. *J. Statist. Phys.*, 44:393–454.

Aizenman, M. and Newman, C. M. (1984). Tree graph inequalities and critical behaviour in percolation models. *J. Statist. Phys.*, 36:107–143.

Cardy, J. L. (1992). Critical percolation in finite geometries. *J. Phys. A*, 25:L201–206.

Chayes, J. T. and Chayes, L. (1987). On the upper critical dimension of Bernoulli percolation. *Comm. Math. Phys.*, 113:27–48.

Fisher, M. E. and Essan, J. W. (1961). Some cluster sizes in percolation. *J. Math. Phys.*, 2(4):609–619.

Goddard, P. and Olive, D. (1988). *Kac-Moody and Virasoro Algebras.* World Scientific.

Grimmett, G. R. (1989). *Percolation.* Springer.

Grimmett, G. R. (1996). Percolation and disordered systems. In *Ecole d'Eté de Probabilités de Saint-Flour XXVI*, pages 153–300. Springer.

Grimmett, G. R. (To be published 1999). *Percolation.* Springer.

Hara, T. (1990). Mean-field critical behaviour for correlation length for percolation in high dimensions. *Probab. Theor. Relat. Fields*, 86:337–385.

Hara, T. and Slade, G. (1990a). Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.*, 128:333–391.

Hara, T. and Slade, G. (1990b). The triangle condition for percolation. *Bull. Amer. Math. Soc. (N.S.)*, 21(2):269–273.

Hara, T. and Slade, G. (1992). Self-avoiding walk in five or more dimensions (I. The critical behavior). *Comm. Math. Phys.*, 147:101–136.

Hara, T. and Slade, G. (1994). Mean-field behaviour and the lace expansion. In *Probability and Phase Transitions*, pages 87–122. Kluwer Academic.

Hughes, B. D. (1996). *Random Walks and Random Environments.* Oxford.

Kesten, H. (1982). *Percolation Theory for Mathematicians.* Birkhäuser.

Langlands, R., Pouliot, P., and Saint-Aubin, Y. (1994). Conformal invariance in two-dimensional percolation. *Bull. Amer. Math. Soc. (N.S.)*, 30(1):1–61.

Langlands, R. P., Pichet, C., Pouliot, P., and Saint-Aubin, Y. (1992). On the universality of crossing probabilities in two-dimensional percolation. *J. Statist. Phys.*, 67:553–574.

Madras, N. and Slade, G. (1993). *The Self-Avoiding Walk.* Birkhäuser.

Nehari, Z. (1952). *Conformal Mapping.* Dover.

Saint-Aubin, Y. (1995). Conformal invariance of a model of percolation on random lattices. *Phys. A*, 221:41–51.

Slade, G. (1987). The diffusion of self-avoiding random walk in high dimensions. *Comm. Math. Phys.*, 110:661–683.

Slade, G. (1991). The lace expansion and the upper critical dimension for percolation. *Lectures in Applied Mathematics*, 27:53–63.

Stauffer, D. (1979). Scaling theory of percolation clusters. *Phys. Rep.*, 54:1–74.

Stauffer, D. (1994). Universality at the three-dimensional percolation threshold. *J. Phys. A*, 27:L474–L480.

Tasaki, H. (1987). Hyperscaling inequalities for percolation. *Comm. Math. Phys.*, 113:49–65.

Ziff, R. M. (1992). Spanning probability in 2D percolation. *Phys. Rev. Lett.*, 69(18):2670–2673.

Ziff, R. M. (1994). Reply to comment on "Spanning probability in 2D percolation". *Phys. Rev. Lett.*, 72(12):1942.

Ziff, R. M. (1995a). On Cardy's formula for the critical crossing probability in 2D percolation. *Phys. A*, 28:1249–1255.

Ziff, R. M. (1995b). Proof of crossing formula for 2D percolation. *Phys. A*, 28:6479–6480.

Ziff, R. M., Cummings, P. T., and Stell, G. (1984). Generation of percolation cluster perimeters by a random walk. *Phys. A*, 17:3009–3017.